Time-Ordered Products in Two-Dimensional Field Theories

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It is shown that the vacuum expectation values (VEV) of $\vdots e^{\lambda\sigma}\vdots(t,f) = \int d\mathbf{x} f(\mathbf{x}) \vdots e^{\lambda\sigma}\vdots(t,\mathbf{x})$ are continuous functions of the time for test functions which are C^∞ and of rapid decrease, with λ in some neighborhood of the origin in the complex plane. The field $\sigma(x)$ is the pseudopotential derived from the pseudovector current of a free two-component massive field in two-dimensional space–time. A consequence of this result is the existence of Green's functions in the Federbush model. An essential technique in the proof is a theorem by Jaffe on the boundary values of limits of sequences of analytic functions.

1. INTRODUCTION

Recent attempts to find examples of nontrivial quantum field theories have led to the study of various two-dimensional models,[1] which, even though the fundamental existence problem remains to be answered, have provided some clear examples of general propositions in field theory. In particular, Wightman has shown how to define a local field $\sigma(f)$ from the bilinear currents of free two-dimensional two-component fields $\psi^{(0)}(f)$, $\psi^{+(0)}(f)$, which has the distinction of being a member of a local equivalence class other than that of the underlying free fields, and also playing a central role in the solution of both the Federbush[2] and massive Thirring models. For the former, a solution of the field equations is given by $\psi(f) = [\vdots e^{\lambda\sigma}\vdots\, \psi^{(0)}](f)$,[3] while as yet no such simple functional form has been found for the latter case. The solution $\psi(f)$ allows an explicit characterization of the perturbation series after a field strength renormalization which is sufficiently compact to lead directly to the existence of the time-ordered vacuum expectation values. Unfortunately for the more interesting Thirring model it appears necessary to deal directly with the Gell-Mann–Low expansion in the manner proposed by Lanford[4] and renormalize by techniques such as those used by Glimm,[5] both for the Yukawa model. The close relation between these two field theories underlines the need for a complete solution to the Federbush model in spite of its rather simple structure.

We proceed by first showing that for any positive integer the vacuum expectation values of $\vdots\sigma^n\vdots\,(t,\mathbf{x})$ are tempered distributions in the space variables with values in the Banach space B^α of bounded functions obeying a Lipschitz condition of order α, $0 < \alpha < 1$, in the time variables. A result of this type was first given by Jaffe[6] for the free field in two-dimensional space–time. The extension to $\vdots e^{\lambda\sigma}\vdots\,(t,\mathbf{x})$ itself, for λ in some neighborhood of the origin, is accomplished by a slight modification of Jaffe's limit theorem[7] on the boundary values of analytic functions. It is clear that this limit theorem may be extended to a variety of countably normed spaces complete with respect to some distribution topology.

Let us recall some definitions which will be needed in our work. The triple-dot ordering of a field is defined by

$$\sigma(x_1)\cdots\sigma(x_n) = \sum_{\text{partitions}} \vdots\sigma(x_{i_1})\cdots\sigma(x_{i_r})\vdots\, \langle\sigma(x_{j_1})\cdots\sigma(x_{j_{n-r}})\rangle_0, \tag{1.1}$$

the sum being taken over all partitions of $(123\cdots n)$ into disjoint subsets $(i_1 i_2\cdots i_r)$, $(j_2 j_2\cdots j_{n-r})$ in natural order. The triple dot of σ to order n is defined by

$$\vdots\sigma^n\vdots\,(x) = \lim_{x_1\ldots x_n\to x} \vdots\sigma(x_1)\cdots\sigma(x_n)\vdots \tag{1.2}$$

and by the reconstruction theorem when this limit exists. It is shown in Ref. 3 that $\vdots\sigma^n\vdots\,(f)$ exists for all

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[1] A. S. Wightman, "Introduction to Some Aspects of the Relativistic Dynamics of Quantized Fields" (I.H.E.S. preprint, 1965).

[2] P. Federbush, Phys. Rev. **121**, 1247 (1961).

[3] J. L. Challifour and A. S. Wightman (unpublished). The relation (2.1) follows from the triple-dot product formula

$$\vdots 1^{m_1}\vdots\,\vdots 2^{m_2}\vdots\cdots\vdots n^{m_n}\vdots = \sum_{\substack{i_j=0\\1\le j\le n}}^{m_j} \frac{\vdots 1^{i_1}2^{i_2}\cdots n^{i_n}\vdots}{i_1!\,i_2!\cdots i_n!}$$
$$\times \sum_{\eta(k_1\cdots k_n)} \prod_{\substack{k_j=0\\1\le j\le n}}^{m_j-i_j} \frac{m_1!\,m_2!\cdots m_n!}{\eta_{k_1\cdots k_n}!}\left[\frac{\langle 1^{k_1}2^{k_2}\cdots n^{k_n}\rangle_0^T}{k_1!\,k_2!\cdots k_n!}\right]^{\eta(k)}.$$

The summation over the nonnegative integers $\eta_{(k_1\cdots k_n)}$ is restricted by

$$\sum_{k_p=0}^{m_p-i_p} k_p \sum_{\substack{k_j=0\\1\le j\le n,\, j\ne p}}^{m_j-i_j} \eta(k_1\ldots k_n) = m_p, \quad 1\le p\le n.$$

This result in turn may be obtained by a series of lengthy, but straightforward induction arguments.

[4] O. Lanford, Thesis, Princeton University, Princeton, N.J., 1966.

[5] J. Glimm, Commun. Math. Phys. **5**, 343 (1967); **6**, 61 (1967).

[6] A. Jaffe, J. Math. Phys. **7**, 1250 (1966).

[7] A. Jaffe, J. Math. Phys. **5**, 1174 (1965).

$f \in \mathcal{S}^8$ as an operator-valued distribution satisfying the usual requirements for a local field.[9] The dense domain in Φ_{0K} space for $\psi^{(0)}(f)$, $\psi^{+(0)}(f)$ is $D = P(\psi^{(0)}, \psi^{+(0)})\Psi_0$, where Ψ_0 is the vacuum state and $\vdots\sigma^n\vdots\,(f)D \subset D, \forall n = 1, 2, 3, \cdots$.

A formal expression for $\sigma(f)$ is given by the convolution

$$\sigma(f) = 2m(\Delta_0 * k)(f), \qquad (1.3)$$

with

$$k(f) = \vdots\psi^{+(0)} i\gamma^5 \psi^{(0)}\vdots\,(f), \quad \Delta_0(x) = \frac{P}{(2\pi)^2}\int dp\,\frac{e^{-ipx}}{p^2}.$$

Our notation and spinor conventions follow Ref. 1:

$$\gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g^{\mu\nu}, \quad g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

This allows us to write

$$\begin{aligned}\langle \sigma(x_1)\cdots\sigma(x_{2N})\rangle_0^T &= (2m)^{2N}\int \prod_{i=1}^{2N} \Delta_0(x_i - y_i) \\ &\quad \times \langle k(y_1)\cdots k(y_{2N})\rangle_0^T\, dy_1\cdots dy_{2N}, \qquad (1.4)\end{aligned}$$

where $\langle\ \rangle_0^T$ indicates the truncated vacuum expectation value. In the next section it is shown that these convolutions exist. Expanding the VEV of the pseudocurrent leads to the formal expression

$$\langle\sigma(x_1)\cdots\sigma(x_{2N})\rangle_0^T = -\left(\frac{im}{\pi}\right)^{2N}\sum_{\pi_{2n}}\int\prod^{2n} \times \frac{dp_i\delta \pm (p_i^2 - m^2)\exp\left[-i\sum_{i=1}^{2n} x_i(p_{\pi(i)} - p_{\pi(i)-1})\right]}{[(p_1 - p_2)^2(p_2 - p_3)^2\cdots(p_{2n} - p_1)^2]^{\frac12}} \qquad (1.5)$$

in which $\theta(\pm p^0_{\pi(i)})$ if $i \lessgtr \pi^{-1}[\pi(i) + 1]$. The summation in (1.5) is over the $(2n-1)!$ cycles of length $2n$ in the symmetric group of order $2n$ with $\pi(1) = 1$, $\pi(l) - 1 = 2n$.

The expressions which are of interest to us are functions of the time variables $\tau = (t_1t_2\cdots t_s)$ defined on $\mathcal{R}^s$ by

$$\omega_s^n(\tau) = \langle \vdots\sigma^{n_1}\vdots\,(t_1, f_1)\vdots\sigma^{n_2}\vdots\,(t_2, f_2)\cdots\vdots\sigma^{n_s}\vdots\,(t_s, f_s)\rangle_0$$

with

$$\vdots\sigma^n\vdots\,(t, f) = \int d\mathbf{x} f(\mathbf{x})\vdots\sigma^n\vdots\,(t, \mathbf{x}), \quad f \in \mathcal{S}.$$

We will not entertain the possibility that f need not be so restricive. Proposition 1 shows that $\omega_s^n(\tau)$ is bounded on $\mathcal{R}^s$, while Proposition 2 verifies that $\omega_s^n(\tau) \in \mathcal{B}^\alpha(\mathcal{R}^s)$, $0 < \alpha < 1$.

2. EXISTENCE OF $\omega_s^n(\tau)$

The structure of $\omega_s^n(\tau)$ is apparent from the product formula[3]

$$\langle\vdots\sigma^{n_1}\vdots\,(x_1)\cdots\vdots\sigma^{n_s}\vdots\,(x_s)\rangle_0 = n!\sum_{n_{(k)}}\prod_{\substack{k=0\\1\le i\le s}}^{n_i}\frac{[\langle\sigma^{k_1}(x_1)\cdots\sigma^{k_s}(x_s)\rangle_0^T/k!]^{n_{(k)}}}{n_{(k)}!} \qquad (2.1)$$

at least two $k_i \neq 0$. Throughout we use Schwartz's[8] multi-index notation. $(k) = (k_1k_2\cdots k_s)$ is a partition of positive integers $0 \le k_s < \infty$, $k! = k_1!\,k_2!\cdots k_s!$, $n! = n_1!\,n_2!\cdots n_s!$ and $n_{(k)}$ are positive integers such that

$$\sum_{\substack{k_i=0\\1\le i\le s}} k_j n_{(k)} = n_j$$

for each $1 \le j \le s$. The truncated VEV inside the bracket is nonzero only if $|k| = k_1 + k_2 + \cdots + k_s$ is even. This is tactitly understood throughout.

For a given partition (k) let $\pi_{(k)}$ be any cycle of length $|k|$ as in (1.5) and write

$$P_j^{\pi(k)} = \sum_{\alpha=k_1+\cdots k_{j-1}+1}^{k_1+\cdots k_j}[p_{\pi(\alpha)} - p_{\pi(\alpha)-1}].$$

Denote by $I^\pi_{(k)}$ that subset of $(123\cdots|k|)$ taken in natural order for which p_β appears in $P_j^{\pi(k)}$ $1 \le j \le s \leftrightarrow \beta \in I^\pi_{(k)}$ for the permutation $\pi_{(k)}$. With this notation we find from (1.5) that, for a given partition (k),

$$\begin{aligned}\langle\sigma^{k_1}(x_1)\sigma^{k_2}(x_2)\cdots\sigma^{k_s}(x_s)\rangle_0^T &= -\left(\frac{m}{\pi}\right)^{|k|}\sum_{\pi(k)}\int\prod_{\alpha\in I^{\pi}(k)} dp_\alpha\delta \pm (p_\alpha^2 - m^2) \\ &\quad \times G^\pi_{(k)}(p_\alpha)\exp\left(-i\sum_{j=1}^s P_j^{\pi(k)}x_j\right), \qquad (2.2a)\end{aligned}$$

in which

$$G^\pi_{(k)}(p_\alpha) = \int\prod_{\beta\in I'^{\pi}_{(k)}}\frac{dp_\beta\delta \pm (p_\beta^2 - m^2)}{[(p_1 - p_2)^2(p_2 - p_3)^2\cdots(p_{|k|} - p_1)^2]^{\frac12}}. \qquad (2.2b)$$

Hence $\omega_s^n(\tau)$ takes the form

$$\begin{aligned}\omega_s^n(\tau) &= n!\sum_{n_{(k)}}\prod_{\substack{k_i=0\\1\le i\le s}}^{n_i}\frac{(-m/\pi)^{|k|n_{(k)}}}{n_{(k)}!\,(k!)^{n_{(k)}}} \\ &\quad \times \sum_{\substack{\pi^r_{(k)}\\1\le r\le n_{(k)}}}\int\prod_{r=1}^{n_{(k)}}\prod_{\alpha\in I^{\pi^r}_{(k)}} dp_\alpha\delta \pm (p_\alpha^2 - m^2) \\ &\quad \times G^{\pi^r}_{(k)}(p_\alpha)\left[\prod_{j=1}^s \tilde{f}_j\left(\sum_{\substack{k_i=0\\1\le i\le s}}^{n_i}\sum_{r=1}^{n_{(k)}}\mathbf{P}_j^{\pi(k)^r}\right)\right] \\ &\quad \times \left\{\exp\left[-i\sum_{j=1}^s t_j\left(\sum_{\substack{k_i=0\\1\le i\le s}}^{n_i}\sum_{r=1}^{n_{(k)}}P_j^{0\pi(k)^r}\right)\right]\right\}. \qquad (2.3)\end{aligned}$$

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[8] L. Schwartz, *Theorie des distributions* (Hermann et Cie., Paris 1957), Tome II.

[9] R. F. Streater and A. S. Wightman; *PCT Spin and Statistics and All That* (W. A. Benjamin, Inc., New York, 1964).

Here $\tilde{f}$ is the Fourier transform of f. Following Jaffe[6] denote $\|f\|_\epsilon = \sup |(1 + |x|)^\epsilon f(x)|$, then we have

Proposition 1:

$$|\omega_s^n(\tau)| \le K_n(\epsilon, s) \prod_{j=1}^{s} \|\tilde{f}_j\|_{\epsilon_j}, \quad \forall \epsilon_j > 0,$$

where $K_n(\epsilon, s)$ is bounded for any finite partition $(n) = (n_1 n_2 \cdots n_s)$.

It is enough to verify that integrals of the form

$$\left|\int \prod_{\substack{k_i=0\\1\le i\le s}}^{n_i} \prod_{r=1}^{n(k)} \prod_{\alpha\in I_{(k)}^{\pi^r}} dp_\alpha \delta \pm (p_\alpha^2 - m^2) \times G_{(k)}^{\pi^r}(p_\alpha)\left[\prod_{j=1}^{s}\left(1 + \left|\sum_{k_i=0}^{n_i}\sum_{r=1}^{n(k)} \mathbf{P}^{\pi(k)^r}\right|\right)^{-\epsilon_j}\right]\right| \qquad (2.4)$$

exist for any partition (k), finite (n) and s, with any $\epsilon = (\epsilon_1, \epsilon_2, \cdots, \epsilon_s) > 0$. $K_n(\epsilon, s)$ is a finite sum of such integrals.

Changing variables to $p_i = m(\cosh\theta_i, \sinh\theta_i)$ $-\infty \le \theta_i \le \infty$ and including all the additional numerical factors, (2.2b) becomes

$$H_{(k)}^{\pi^r}(\theta_\alpha) = \frac{1}{(4m)^{|k|}} \int_{-\infty}^{\infty} \prod_{\beta\in I'^{\pi^r}_{(k)}} d\theta_\beta \times g_1(\theta_1 - \theta_2)(g_2(\theta_2 - \theta_3) \cdots g_{|k|}(\theta_{|k|} - \theta_1),$$

where

$$g_i(\theta) = \begin{cases} \sinh^{-1}\theta/2 & \theta(p_i^0 p_{i+1}^0), \\ \cosh^{-1}\theta/2 & \theta(-p_i^0 p_{i+1}^0). \end{cases}$$

For a given r $I_{(k)}^{\pi^r} = \{\alpha_1, \alpha_2, \cdots \alpha_{\lambda r}\}$ in natural order, so that

$$H_{(k)}^{\pi}(\theta_\alpha) = h_{\alpha_1}(\theta_{\alpha_1} - \theta_{\alpha_2}) h_{\alpha_2}(\theta_{\alpha_2} - \theta_{\alpha_3}) \cdots h_{\alpha_\lambda}(\theta_{\alpha_{\lambda_r}} - \theta_{\alpha_1}), \qquad (2.5)$$

where each $h_i(\theta)$ is given by a chain of length $l_i = \alpha_{i+1} - \alpha_i - 1$ consisting of the convolutions

$$h_{\alpha_i}(\theta) = [g_{\alpha_i} * g_{\alpha_i+1} * \cdots * g_{\alpha_{i+1}-1}](\theta). \qquad (2.6)$$

Each $g_i(\theta)$ is a distribution in O'_C, being the Fourier transform of the O_M function $i\tanh\pi\sigma$, or else lies in $\mathcal{S}$, being the Fourier transform of $\cosh^{-1}\pi\sigma$. Further, if any g_i in the chain (2.6) lies in $\mathcal{S}$, then so does the whole convolution. Otherwise $h_{\alpha_i}(\theta) \in O'_C$.[10]

Moreover, for any chain of length l_i, $0 \le l_i < \infty$, $h_{\alpha_i}(\theta)$ varies over bounded sets in $\mathcal{S}$ and O'_C, respectively. In the case that $h_{\alpha_i}(\theta) \in O'_C$ and l_i is odd, $h_{\alpha_i}(\theta)$ may be written as a sum of functions in $\mathcal{S}$ together with $\sinh^{-1}\theta/2$; while for l_i even the chain is again a sum of functions in $\mathcal{S}$ but now the distribution is $\delta(\theta)$. As a last remark on the properties of (2.5) let us note that for any partition (k) and permutation $\pi_{(k)}^r$ at least one term in (2.5) is in $\mathcal{S}$. This is a consequence of the observation that $g_{|k|}(\theta) = \cosh^{-1}\theta$ for any such permutation.

[10] Reference 8, Théorèmes XI et XV, pp. 103, 124.

In terms of these variables, (2.4) becomes a repeated application of integrals of the form

$$P\int_{-\infty}^{\infty} \prod_{\alpha\in I_{(k)}^{\pi^r}} d\theta_\alpha h_{\alpha_1}(\theta_{\alpha_1} - \theta_{\alpha_2}) \cdots h_{\alpha_{n-1}}(\theta_{\alpha_{n-1}} - \theta_{\alpha_n}) \times h_{\alpha_n}(\theta_{\alpha_n} - \theta_{\alpha_1}) f(z + u(\theta)), \qquad (2.7)$$

where $z + u(\theta) = \{z_1 + u_1(\theta), \cdots, z_s + u_s(\theta)\}$ and

$$u_j(\theta) = 1 + m^2\left(\sum_{\substack{k_i=0\\1\le i\le s}}^{n_i} \sum_{r=1}^{n(k)} \sum_{\alpha=k_1+\cdots+k_{j-1}+1}^{k_1+\cdots+k_j} \times \{\epsilon^r_{\pi(k)}(\alpha)\sinh\theta_{\pi^r_{(k)}(\alpha)} - \epsilon^r_{\pi(k)(\alpha)-1}\sinh\theta_{\pi^r_{(k)}(\alpha)-1}\}\right)^2,$$

ϵ_π being the sign for the permutation π. Here $f(z) \in C_0^\infty(\mathcal{R}^s)$ with

$$|D^p f(z)| \le K \prod_{i=1}^{s} [1 + z_i^2]^{-(p_i+\epsilon_i)/2} (p) = (p_1, \cdots, p_s), \quad (\epsilon) = (\epsilon_1, \cdots, \epsilon_s) > 0, \quad \text{for } |z$$

large enough. It is convenient to relabel the variables and transform again by setting $\theta_{\alpha_i} = x_i + x_{i+1} + \cdots + x_n$ so that (2.7) becomes

$$I^{(p)}(z) = P\int_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i h_1(x_1) h_2(x_2) \cdots h_{n-1}(x_{n-1}) \times h_n\left(+\sum_{i=1}^{n-1} x_i\right) D^p f(z + v(x)), \qquad (2.8a)$$

where $D^p = \partial^{|p|}/\partial^{p_1} z_1 \cdots \partial^{p_s} z_s$ and $v(x)$ is $u(\theta)$ expressed in terms of the new variables. The relabeling is chosen so that $|h_n(x)| \le \cosh^{-1} x/2$ while the remaining $h_i(x)$ are either test functions in $\mathcal{S}$ or distributions $\delta(x)$, $\sinh^{-1} x/2$. It is clearly enough to take each $h_i(x) = \sinh^{-1} x/2$ $1 \le i \le n-1$ in (2.8). The alternative cases are dealt with similarly.

We then study the integral

$$D^{(p)} I(z) = P\int_{-\infty}^{\infty} \prod_{i=1}^{n} dx_i \sinh^{-1}\frac{x_1}{2} \cdots \sinh^{-1}\frac{x_{n-1}}{2} \times h_n\left(+\sum_{i=1}^{u-1} x_i\right) D^p f(z + v(x)) \qquad (2.8b)$$

and show the following result.

Lemma 1: Let f be C^∞ and in Weinberg's class $\mathcal{F}_\mu^\infty(\mathcal{R}^s)$ where $\mu = \min(\epsilon_i)$, $0 < \mu < 1$. Then so is the integral $I(z)$.

Proof: The proof is an application of Weinberg's asymptotic theorem[11] after the region of integration

[11] S. Weinberg, Phys. Rev. **118**, 838 (1960).

has been suitably partitioned to take into account the singular nature of the integrand. For the definitions of a multidimensional singular integral we refer the reader to Dunford and Schwartz.[12]

Let $(i)_m = (i_1 i_2 \cdots i_m)$ $1 \leq m \leq n-1$ be any subset of m distinct elements in natural order chosen from $(123 \cdots n-1)$. Then we define an $(i)_m - \delta$ corridor by

$$Q_{(i)_m}(\delta) = Q_{i_1 i_2 \cdots i_m}(\delta) = \{x \mid |x_i| \leq \delta \; i \in (i)_m, \; |x_j| > \delta \; j \notin (i)_m, \; j \neq n\}.$$

The part of the corridor with $|x_i| \leq \delta$ $i \in (i)_m$ we shall refer to as an $(i)_m - \delta$ box. Then the region of integration in (2.8b) is partitioned into disjoint sets by

$$\mathcal{R}^n = \left[\bigcup_{1 \leq m \leq n-1} Q_{(i)_m}(\delta)\right] \cup \mathcal{R}_0^n.$$

The nonsingular region may also be written $\mathcal{R}_0^n = \bigcup_{(j)} \omega_{(j)}^n$ where $\omega_{(j)}^n$ is a wedge defined by the coordinate planes; i.e., consider a set of signs

$$x_1 \gtrless 0 \quad x_2 \gtrless 0 \cdots x_n \gtrless 0,$$

2^n in number. Then $1 \leq j \leq 2^n$ and $\omega_{(j)}^n$ is the corresponding region defined by a particular j.

We now consider the behavior of $I(z)$ on each of these disjoint regions.

Wedges: On each wedge the integrand is bounded and

$$|I(\omega_j^n)| \leq \int \prod_{i=1}^n dx_i \, |f(z + v(x))| \leq \int_{\mathcal{R}^n} \prod_{i=1}^n \frac{dp_i}{(m^2 + p_i^2)^{\frac{1}{2}}} |f(z + \Sigma \epsilon p_i)|.$$

Weinberg's asymptotic theorem and Jaffe's analysis[6] gives the existence of the integral and that it belongs to $\mathcal{F}_\mu^\infty$, $0 < \mu < 1$.

Corridors: On a corridor $Q_{(i)_m}(\delta)$ with the notation

$$\Delta_{(i)_m} = \Delta_{i_1} \Delta_{i_2} \cdots \Delta_{i_m},$$

$$\Delta_j h(x_1 \cdots x_n) = h(x_1, \cdots, x_j, \cdots x_n) - h(x_1, \cdots, -x_j, \cdots, x_n),$$

the integral becomes the limit

$$I(Q_{(i)_m}(\delta)) = \lim_{(\epsilon)_{(i)_m} \to 0^+} \int_{\epsilon_{i_1}}^{\delta} \frac{dx_{i_1}}{\sinh x_{i_1}/2} \cdots \int_{\epsilon_{i_m}}^{\delta} \frac{dx_{i_m}}{\sinh x_{i_m}/2} \times \int_{|x_j| > \delta} \prod_{j=1}^{n-1} \frac{dx_j \, dx_u}{\sinh x_j/2} \times \Delta_{(i)_m}\left[h_n\left(+\sum_{i=1}^{n-1} x_i\right) D^\nu f(z + v(x))\right]. \tag{2.9}$$

[12] N. Dunford and J. Schwartz, *Theory of Linear Operators* (Interscience Publishers, Inc., New York, 1965), Vol. II.

The unbounded part of the corridor may be broken into compact and unbounded regions

$$\delta < |x_j| \leq N_j, \quad |x_j| > N_j, \quad j \notin (i)_m.$$

On the compact regions, the C^∞ nature of $f(z)$ and $h_n(x)$ together with the boundedness for large (z) permit the estimate

$$|\Delta_{(i)_m}\{h_n(x_1 + \cdots + x_{n-1}) D^\nu f[z + v(x)]\}| \leq 2^m x_i \cdots x_{i_m} \times \left|\frac{\partial^m}{\partial x_{i_1} \cdots \partial x_{i_m}}\left\{h_n\left(\sum_{i=1}^{n-1} x_i\right) D^\nu f[z + v(x)]\right\}\right|_{x_{i_1}^* \cdots x_{i_m}^*}, \tag{2.10}$$

with $x_{i_1}^* \cdots x_{i_m}^*$ some point in the $(i)_m - \delta$ box. Weinberg's asymptotic theorem together with his equation (12) for the asymptotic coefficients shows that $I^\nu[Q_{(i)_m}(\delta)]$ in (2.9) over the compact regions is again in $\mathcal{F}_\mu^\infty(\mathcal{R}^s)$.

For the unbounded regions choose $N_j \, j \notin (i)_m, j \neq n$, such that for $|x_j| > |N_j|$,

$$\frac{1}{|\sinh x_j/2|} \leq \frac{K_M |x_{i_1} \cdots x_{i_m}|^\gamma}{\prod_{i=1}^s [1 + z_i^2]^{(\epsilon_i + p_i)/2} [1 + x_j^2]^{M_j/2}},$$

$$0 < \gamma < 1, \quad M_j \text{ a large integer.}$$

This may be achieved for any z, (p), (ϵ). The regions under consideration break down into three types:

(i) $\delta < |x_j| \leq N_j$, $j \neq n$, $|x_n| > N_n$;
(ii) $|x_n| \leq N_n$ with at least one $|x_j| \geq N_j$, $j \neq n$, $j \notin (i)_m$;
(iii) $|x_n| > N_n$ and at least one $|x_j| \geq N_j$ as in (ii).

Case (ii) is readily handled. $\forall(\epsilon)$, (p), $z \; \exists \; K_m$, $(M) = (M_1 \cdots M_j \cdots M_{n-1})$, $0 < \gamma < 1$ such that for $|z|$ large enough, the integral is bounded by

$$\lim_{(\epsilon)_{(i)_m} \to 0^+} \int_{\epsilon_{i_1}}^{\delta} \frac{dx_{i_1} x_{i_1}^\gamma}{\sinh x_{i_1}/2} \cdots \int_{\epsilon_{i_m}}^{\delta} \frac{dx_{i_m} x_{i_m}^\gamma}{\sinh x_{i_m}/2} \times \int \prod_{j=1}^{n-1}{}' \, dx_j \frac{2K_M N_n C}{(1 + x_j^2)^{M_j/2} \prod_{i=1}^s [1 + z_i^2]^{(\epsilon_i + p_i)/2}},$$

where

$$C = \sup_{\mathcal{R}^{(n+s)}} |\Delta_{(i)_m}\{h_n(x_1 + \cdots x_{n-1}) D^\nu f[z + v(x)]\}|.$$

This integral exists as $(\epsilon)_{(i)_m} \to O^+$ and satisfies the estimates at large $|z|$ for $\mathcal{F}_\mu^\infty$. Over the region of integration appropriate to Case (i) let us write

$$\Delta_{(i)_m}\{h_n(x_1 + \cdots + x_{n-1})D^\nu f[z + v(x)]\} = \sum_{\substack{m_1+m_2=m \\ \text{partitions}}} [\Delta_{(i)_{m_1}} h'_n(x_1 + \cdots + x_{n-1})] \times \Delta_{(i)_{m_2}} D^\nu f[z + v(x)]\}$$

[$'$ means $-x_{i_k}$ if $k \in (i)_{m_2}$]. As $h_n \in \mathcal{S}$ and each x_j $1 \le j \le n-1$ varies over a compact, we may estimate $\Delta_{(i)_{m_1}} h'_n$ terms by the mean-value theorem as $2^{m_1}K\left|\prod_{k\in(i)_{m_1}} x_k\right|$. The $\Delta_{(i)_{m_2}} D^\nu f(z + v(x))$ terms are a little more delicate. The case $m_2 = 0$ reduces to the discussion for wedges $\omega_j^{(n)}$. When $m_2 \neq 0$ let us examine one difference:

$$\Delta_{i_k} D^\nu f[z + v(x)] = D^\nu f[z + v(x_1 \cdots x_{i_k} \cdots x_n)] - D^\nu f[z + v(x_1 \cdots -x_{i_k} \cdots x_n)].$$

Recalling that each $\theta_\alpha = x_\alpha + x_{\alpha+1} \cdots + x_n$, we may choose N_n sufficiently large that the Cauchy convergence condition gives

$$|\Delta_{(i)_{m_2}} D^\nu f[z + v(x)]| \le \frac{K \prod_{j\in(i)_{m_2}} |x_j|^\gamma}{\prod_{i=1}^{s} [1 + (z_i + v_i)^2]^{(\epsilon_i+p_i)/2}}, \qquad 0 < \gamma < 1,$$

for $|x_n| > N_n$. Weinberg's theorem is now applicable to this estimate to show the existence of the integral as $(\epsilon)_{(i)_m} \to O^+$ with the correct asymptotic behavior at large $|z|$ for $\mathcal{F}_\mu^\infty$.

Case (iii) is clearly a compounding of cases (i) and (ii) with no new conditions arising.

This concludes the proof of the lemma.

An application of Lemma 1 leads directly to the existence of (2.4), as these integrals are a repeated iteration of integrals of the form (2.8a) with $(z) = 0$ at the last stop. The order of these integrals is also immaterial since the integrals are absolutely convergent in any order.

From Proposition 1, we have seen the existence of $\omega_s^n(\tau)$ as a bounded function on $\mathfrak{R}^s$. The continuity in the time variables is a direct consequence of simple inequalities.

Proposition 2: $\omega_s^n(\tau) \in B^\alpha(\mathfrak{R}^s)$ for any finite (n) and some $0 < \alpha < 1$.

Proof: $B^\alpha(\mathfrak{R}^s)$ is complete with respect to the norm

$$\|\omega_s^n(\tau)\|^\alpha = \sup \|\omega_s^n(\tau)\| + \sup_{\tau \neq \tau'} \frac{|\omega_s^n(\tau) - \omega_s^n(\tau')|}{\|\tau - \tau'\|^\alpha},$$

where

$$\|\tau - \tau'\|^\alpha = \left[\sum_{i=1}^{s} (t_i - t'_i)^2\right]^{\alpha/2}$$

To verify the existence of the second supremum, the identical theorem to Jaffe's[6] equations (32) and (33) holds in this case, to give

$$|\omega_s^n(\tau) - \omega_s^n(\tau')| \le \sum_{k=1}^{s} K_k^n(\epsilon, s)\, |t_k - t'_k|^\alpha, \quad 0 < \alpha < 1$$

uniformly on $\mathfrak{R}^s$. If $K^n = \max[K_1^n, K_2^n, \cdots, K_s^n]$, then the elementary inequality $\sum_{k=1}^{s} |t_k - t'_k|^\alpha \le 2^{s(1-\alpha/2)}\,\|t - t'\|^\alpha$ gives

$$|\omega_s^n(\tau) - \omega_s^n(\tau')| \le K2^{s(1-\alpha/2)}\,\|\tau - \tau'\|^\alpha$$

and thus $\omega_s^n(\tau) \in B^\alpha(\mathfrak{R}^s)$.

3. EXPONENTIAL

Our next consideration concerns demonstrating that $\omega_s^\infty(\tau) \in B^\alpha(\mathfrak{R}^s)$, where, for any test functions $f_i \in \mathcal{S}$,

$$\omega_s^\infty(\tau) = \langle :e^{\lambda\sigma}: (t_1, f_1) :e^{\lambda\sigma}: (t_2, f_2) \cdots :e^{\lambda\sigma}: (t_s, f_s)\rangle_0. \tag{3.1}$$

More precisely, let us define the operator-valued distributions

$$\phi_\lambda^N(x) = \sum_{n=0}^{N} \frac{\lambda^n :\sigma^n: (x)}{n!}$$

and their vacuum expectation values

$$\omega_s^N(\xi_1, \xi_2, \cdots, \xi_{s-1}) = \langle \phi_{\lambda_1}^N(x_1)\phi_\lambda^N(x_2) \cdots \phi_\lambda^N(x_s)\rangle_0. \tag{3.2}$$

The completeness of $B^\alpha(\mathfrak{R}^s)$ would allow us to conclude that $\omega_s^\infty(\tau)$ is bounded and satisfies a Lipschitz condition provided that we could show the sequence $\{\omega_s^N(\tau)\}$ constructed from (3.2) to be Cauchy with respect to the norm $\|\ \|^\alpha$. As it stands, a direct estimate of this condition is fairly hard due to the detailed nature of the series (2.3). A more natural method of proof is suggested by Jaffe's limit theorem[7] since (3.2) has a formal sum as $N \to \infty$, which may be used in the tube $\mathcal{C}_s = \mathfrak{R}^{2(s-1)} - iV_+^{\otimes(s-1)}$. In fact, the $\omega_s^N(\xi)$ in (3.2) are boundary values of functions $F_s^N(\zeta)$, $\zeta = \xi - i\eta$ analytic in $\mathcal{C}_s$. In a similar manner the $\omega_s^N(\tau)$ are boundary values of functions $H_s^N(\zeta^0)$ analytic in $\Pi^{s-1} = \mathfrak{R}^{s-1} - i\mathfrak{R}_+^{s-1}$ ($\mathfrak{R}_+^s$ is the set of vectors $(x_1, x_2, \cdots, x_s)$ with positive real components), where, in the sense of $\mathcal{S}'$,

$$H_s^N(\zeta^0) = \lim_{\substack{\eta\to O^+ \\ \text{in } V_+}} \int d\mathbf{x}_1 \cdots d\mathbf{x}_s f_1(\mathbf{x}_1) \cdots f_s(\mathbf{x}_s) F_s^N(\zeta). \tag{3.3}$$

The various relations between these sequences of analytic functions and the convergence of their boundary values is contained in the diagram

$$\begin{array}{ccc} F_s^N(\zeta) & \to & F_s^\infty(\zeta), \\ \downarrow & & \downarrow \\ H_s^N(\zeta^0) & \to & H_s^\infty(\zeta^0), \\ \downarrow & & \downarrow \\ \omega_s^N(\tau) & \to & \omega_s^\infty(\tau), \end{array}$$

where the regions in which the limits exist must be specified. In particular, we show that the compact convergence of $F_s^N(\zeta) \to F_s^\infty(\zeta)$ as analytic functions in the tube $\mathcal{C}_s$ leads to $\omega_s^N(\tau) \to \omega_s^\infty(\tau)$ as $N \to \infty$ in the sense of B^α.

To simplify the notation, let us write for (1.5)

$$G_s^{(k)}(\xi) = \langle \sigma^{k_1}(x_1)\sigma^{k_2}(x_2)\cdots\sigma^{k_s}(x_s)\rangle_0^T,$$

where $(k) = (k_1, k_2, \cdots, k_s)$ is any partition with $|k|$ even. Then in the tube $\mathcal{C}_s$ consider the sum

$$F_s^N(\zeta) = \sum_{\substack{n_i=0 \\ 1\le i\le s}}^{N} \lambda_1^{n_1}\lambda_2^{n_2}\cdots\lambda_s^{n_s} \sum_{\eta(k)} \prod_{\substack{k_i=0 \\ 1\le i\le s}}^{n} \frac{[G_s^{(k)}(\zeta)/k!]^{\eta(k)}}{\eta_{(k)}!} \quad (3.4)$$

whose boundary value is the distribution $\omega_s^N(\xi)$. $G_s^{(k)}(\zeta)$ is analytic in $\mathcal{C}_s$ as may be seen directly from (1.5) and the remarks in the Appendix. Further, on any compact subset $K \subset \mathcal{C}_s$,

$$|F_s^N(\zeta)| \le \exp\left[\sum_{(k)} \frac{|\lambda|^k |G_s^{(k)}(\zeta)|}{k!}\right], \quad (3.5)$$

in which the summation over the partition (k) must have at least two nonzero entries, $|\lambda|^k = |\lambda_1|^{k_1}|\lambda_2|^{k_2}\cdots|\lambda_s|^{k_s}$. In the Appendix it is shown that for $\zeta = \xi - it\eta$, $0 < t < 1$, and η in a compact of $V_+^{\otimes s}$,

$$|G_s^{(k)}(\zeta)| \le \frac{(|k|-1)!}{2^{|k|}\pi} K_0[\ln 1/t + K_0']. \quad (3.6)$$

The $(|k|-1)!$ arises from the number of terms in (1.5), each of which is analytic in $\mathcal{C}_s$. Using this in (3.5), noting that for $\zeta \in K$, $\exists$ a t_0 such that $0 < t_0 \le t < 1$ for which the uniform bound $|G_s^{(k)}(\zeta)| \le (|k|-1)!\, M$ holds, and after performing the summation over (k), we have

$$\sum_{(k)} \frac{|\lambda|^k |G_s^{(k)}(\zeta)|}{k!} \le \frac{M}{2} \ln \frac{(1-|\lambda_1|^2)(1-|\lambda_2|^2)\cdots(1-|\lambda_s|^2)}{1-(|\lambda_1|+\cdots+|\lambda_s|)^2}. \quad (3.7)$$

Thus, we may find some neighborhood $N_\epsilon(0)$ of the origin in the $(\lambda) = (\lambda_1, \cdots, \lambda_s)$ complex space such that $F_s^N(\zeta)$ converges compactly to a function $F_s^\infty(\zeta)$ analytic in $N_\epsilon(0) \times \mathcal{C}_s$[13]:

$$\lim_{N\to\infty} F_s^N(\zeta) = F_s^\infty(\zeta) = \exp\left(\sum_{(k)} \frac{\lambda^k G_s^{(k)}(\zeta)}{k!}\right). \quad (3.8)$$

From this last statement, it is an easy matter to verify that $H_s^N(\zeta^0)$ converges compactly to $H_s^\infty(\zeta^0)$, analytic in $N_\epsilon(0) \times \Pi_-^{s-1}$ where

$$H_s^\infty(\zeta^0) = \lim_{\substack{\eta\to\infty \\ \text{in } V_+}} \int dx_1 \cdots dx_s f_1(x_1)\cdots f_s(x_s) F_s^\infty(\zeta). \quad (3.9)$$

To complete the diagram above, we now give a lemma on the boundary values of functions analytic in Π_-^s.

Lemma 2: Let $\Phi(\xi)$ be analytic in Π_-^s. Then $\Phi(\xi)$ has a boundary value $\Psi(\xi) = \lim_{\eta\to O+} \Phi(\xi - i\eta)$ in

$$B^\alpha(\mathcal{R}^s) \leftrightarrow |\partial\Phi(\zeta)/\partial\zeta_i| \le K\, \|\zeta - \xi_0\|^{\alpha-1}, \quad 1 \le i \le s,$$

$0 < \alpha < 1$ as $\zeta \to \xi_0$ with $\eta \to O^+$ and ζ remaining inside any compact set $\Lambda \subset \Pi_-^{s-1}$ which only meets the boundary $\mathcal{R}^s$ in the point ξ_0, which may be arbitrary.

Proof: The necessity of the above condition is a simple generalization to s variables of Privalov's lemma.[14]

For the sufficiency, consider

$$|\Phi(\xi - it\eta) - \Phi(\xi - it'\eta)| \le \int_{t'}^{t} d\tau \left|\frac{\partial}{\partial\tau}\Phi(\xi - i\tau\eta)\right|,$$

where $\xi \in \mathcal{R}^s$ and η varies over compacts in $\mathcal{R}_+^s$; $0 < t < 1$. The hypotheses of the lemma allows the estimate

$$|\Phi(\xi - it\eta) - \Phi(\xi - it'\eta)| \le \frac{K}{\alpha}\left[\sum_{i=1}^{s} |\eta_i|^\alpha\right](t^\alpha - t'^\alpha)$$

so that $\{\Phi(\xi - it\eta)\}$ is a Cauchy sequence of bounded continuous functions as $t \to O^+$ with respect to the supremum norm. Thus $\lim_{t\to 0} \Phi(\xi - it\eta) = \Psi(\xi)$ exists and is continuous and bounded on $\mathcal{R}^s$.

Next consider ξ, ξ' on the boundary $\mathcal{R}^s$. We may choose $0 < t < 1$ and η in a compact of $\mathcal{R}_+^s$ so that

$$\begin{aligned} |\Psi(\xi) - \Psi(\xi')| &\le |\Psi(\xi) - \Phi(\xi - it\eta)| + |\Phi(\xi - it\eta) - \Phi(\xi' - it\eta)| \\ &\quad + |\Phi(\xi' - it\eta) - \Psi(\xi')|, \end{aligned}$$

[13] The uniformity on compacts is a consequence of Vitali's theorem [E. C. Titchmarsh, *Theory of Functions* (Oxford University Press, New York, 1939), 2nd ed., paragraph 5.21].

[14] N. J. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

where we have just seen that the first and third terms are bounded by $k/\alpha[\sum_{i=1}^{s}|\eta_i|^\alpha]t^\alpha$. For the second term we have

$$|\Phi(\xi - it\eta) - \Phi(\xi' - it\eta)|$$
$$\leq K\sum_{i=1}^{s}\int_{\xi_i'}^{\xi_i} d\rho_i\left[\sum_{k=1}^{i-1}(\xi_k - \xi_{k0})^2 + \sum_{k=i+1}^{s}(\xi_k - \xi_{k0})^2 + (\rho_i - \xi_{i0})^2 + t^2\sum_{k=1}^{s}|\eta_k|^2\right]^{(\alpha-1)/2},$$
$$\text{with} \quad \xi_i' < \xi_{i0} < \xi_i,$$

the right-hand side being bounded by

$$\frac{2^{1-\alpha}K}{\alpha}\sum_{i=1}^{s}|\xi_i - \xi_i'|^\alpha.$$

Finally, we may find a constant K_0 independent of ξ, ξ' for which

$$|\Psi(\xi) - \Psi(\xi')| \leq K_0\,\|\xi - \xi'\|^\alpha, \quad 0 < \alpha < 1,$$
$$t = \min(1, \|\xi - \xi'\|).$$

This last lemma provides a convenient criterion for deducing when the boundary values of $H_s^\infty(\zeta^0)$ belong to B^α. It remains only to be able to decide when such boundary values are actually the limits $\omega_s^\infty(\tau)$ with respect to $\|\ \|^\alpha$, and then apply the reconstruction theorem to recover the field $:e^{\lambda\sigma}:(f)$. This is provided by Lemma 3 below which is quoted without proof. This is identical to the one given in Ref. 7.

Lemma 3: Let $\Phi_N(\zeta)$, $N = 0, 1, 2, \cdots$ be a sequence of functions analytic in Π_-^s and bounded on compacts independently of N, such that $|\partial\Phi_N(\zeta)/\partial\zeta_i| \leq K\,\|\zeta - \xi_0\|^{\alpha-1}$ $1 \leq i \leq s$, $0 < \alpha < 1$ for $\zeta \to \xi_0$ under the same conditions as in Lemma 2. Then

$$\lim_{\eta\to O^+}\Phi_N(\xi - i\eta) = \Psi_N(\xi) \in B^\alpha(\mathfrak{R}^s)$$

and $\{\Psi_N(\xi)\}$ varies over a bounded set in B^α. Moreover, if $\Phi_N(\zeta) \to \Phi_\infty(\zeta)$ pointwise in Π_-^s, then $\Psi_N(\xi) \to \Psi_\infty(\xi)$ in B^α uniformly on bounded sets.

This last result indicates that in order to show $\omega_s^\infty(\tau) \in B^\alpha$, it is only necessary to check the growth condition $|\partial H_s^N(\zeta^0)/\partial\zeta_i| \leq K\,\|\zeta^0 - \tau\|^{\alpha-1}$, $1 \leq i \leq s$, $0 < \alpha < 1$, where τ is some real point in $\mathfrak{R}^s$. The condition on the manner in which ζ^0 approaches the point τ allows us to find a cone with angles $0 \leq \delta_i < \pi$, $1 \leq i \leq s$ such that $\zeta^0 \to \tau$ inside this cone and $\zeta^0 - t_i = (\xi_i^0 - t_i) - i\eta_i = t\eta_i[\tan\omega_i - i]$, where $0 < t < 1$; η_i^0 varying over a compact in $\mathfrak{R}_+$. As $\zeta_i^0 \to t_i$ the angle ω_i is strictly less than $\pi/2$ in magnitude. (See Fig. 1.) Then

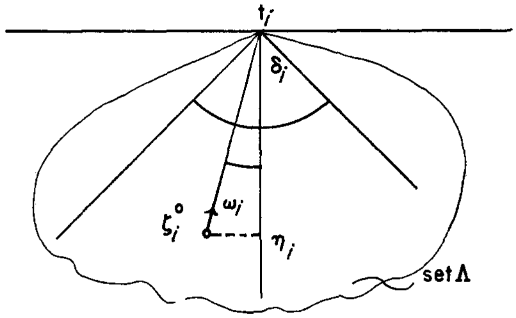

FIG. 1. Region such that $\zeta^0 \to \tau$ inside cone defined by $0 \leq \delta_i \sim \pi$.

$$|\zeta^0 - \tau| = \left(\sum_{i=1}^{s-1}|\zeta_i^0 - t_i|^2\right)^{\frac{1}{2}}$$
$$= t\left[\frac{\eta_1^{0^2}}{\cos^2\omega_1} + \cdots + \frac{\eta_{s-1}^{0^2}}{\cos^2\omega_{s-1}}\right]^{\frac{1}{2}},$$

the coefficient of t being bounded for $\zeta^0 \to \tau$ in the cone and η^0 varying over compacts of $\mathfrak{R}_+^{s-1}$.

With these preliminaries it is now necessary to estimate the growth of the derivatives of $H_s^N(\zeta^0)$ as t approaches zero. For this, we use the bound below obtained from (3.3), (3.4), and (3.5):

$$\left|\frac{\partial H_s^N(\zeta^0)}{\partial\zeta_i^0}\right| \leq \lim_{\substack{\eta\to 0\\ mV_+}}\left|\int d\mathbf{x}_1\cdots d\mathbf{x}_s f_1(\mathbf{x}_1)\cdots f_s(\mathbf{x}_s)\left[\sum_{(k)}\frac{|\lambda|^k}{k!}\left|\frac{\partial G_s^{(k)}(\zeta)}{\partial\zeta_i^0}\right|\right] \times \exp\left[\sum_{(k}\frac{|\lambda|^k}{k!}|G^{(k)}(\zeta)|\right]\right|. \quad (3.10)$$

From the Appendix we find that in the limit $\eta \to 0$ in $V_+^{\otimes s}$, $\zeta^0 = \xi^0 - it\eta^0$, $0 < t < 1$,

$$\left|\int d\mathbf{x}_1\,d\mathbf{x}_2\cdots d\mathbf{x}_s f_1(\mathbf{x})\cdots f_s(\mathbf{x}_s)\frac{\partial G_s^{(k)}(\zeta)}{\partial\zeta_i^0}\right| \leq \frac{t^{\epsilon-1}M\,|k|!}{2^{|k|}\pi}\left[(\ln 1/t)^{|k|-1} + K_0'\right]. \quad (3.11)$$

Here $0 < \epsilon < 1$, while M, K_0' are constants. Lastly, before using these estimates in (3.10), let us note that for every $|k| > 0$, $(\ln 1/t)^{|k|} = 0[1/t^\mu]$ with any $0 < \mu < 1$. This remark together with (3.10) leads directly to the bound

$$\left|\frac{\partial H_s^N(\zeta^0)}{\partial\zeta_i^0}\right| \leq \frac{MN_0t^{\epsilon-1}}{\pi\ln(1/t)}t^{-\mu-r}, \quad (3.12)$$

where

$$N_0 = \frac{\frac{1}{4}(|\lambda_1| + \cdots + |\lambda_s|)^2}{1 - \frac{1}{4}(|\lambda_1| + \cdots + |\lambda_s|)^2} - \frac{\frac{1}{4}|\lambda_1|^2}{1 - \frac{1}{4}|\lambda_1|^2} - \cdots - \frac{\frac{1}{4}|\lambda_s|^2}{1 - \frac{1}{4}|\lambda_s|^2},$$

$$r = \frac{K}{2\pi}\ln\left[\frac{(1 - \frac{1}{4}|\lambda_1|^2)\cdots(1 - \frac{1}{4}|\lambda_s|^2)}{1 - \frac{1}{4}(|\lambda_1| + \cdots + |\lambda_s|)^2}\right],$$

and M is some appropriate constant. If we restrict (λ) to a neighborhood of the origin in which $|\lambda_i| < 2\delta/s$, then $r < (K/2\pi) \ln (1/1 - \delta^2)$. In particular, if $\delta < (\epsilon'/\epsilon' + K/2\pi)^{\frac{1}{2}}$, $0 < \epsilon' < 1$, then $r < \epsilon'$.

Combining these remarks, we may state finally that $\exists\, 0 < \epsilon' < \alpha < 1$ for any $0 < \epsilon' < 1$ such that

$$\left|\frac{\partial H_s^N(\zeta^0)}{\partial \zeta_i^0}\right| \leq K t^{\alpha-1}, \quad 0 < t < 1, \quad 1 \leq i \leq s-1$$

uniformly in some neighborhood $N_\epsilon(0)$ of the origin in the (λ) plane. By Lemma 3 we then have the result below.

Proposition 3: For any test functions $f_i \in \mathcal{S}$, there exists some neighborhood $N_\epsilon(O)$ of the origin in the λ complex plane such that

$$\omega_s^\infty(\tau) = \langle :e^{\lambda\sigma}: (t, f_1) \cdots :e^{\lambda\sigma}: (f_s, t_s)\rangle_0$$

is in $B^\alpha(\mathcal{R}^s)$, $0 < \alpha < 1$ for all $\lambda \in N_\epsilon(0)$.

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APPENDIX: GROWTH ESTIMATES FOR $G_s^{(k)}(\zeta - i\eta)$

During the discussion of $\omega_s^\infty(\tau)$ in Sec. 3, two estimates [(3.6), (3.11)] on the growth of $G_s^{(k)}(\xi - it\eta)$ as $t \to O^+$ were required. We give an account of these below.

From (2.2a) we may start with the formula

$$G_s^{(k)}(\xi - it\eta) = \left(-\frac{m}{\pi}\right)^{|k|} \sum_{\pi(k)} \int \prod_{i=1}^{|k|} \times \frac{dp_i \delta \pm (p_i^2 - m^2) \exp\left[-i \sum_{j=1}^{s-1} \tilde{P}_j^{\pi(k)}(\xi_j - it\eta_j)\right]}{[(p_1 - p_2)^2 (p_2 - p_3)^2 \cdots (p_{|k|} - p_1)^2]^{\frac{1}{2}}}, \tag{A1}$$

where

$$\tilde{P}_j^{\pi(k)} = \sum_{\alpha=1}^{k_1 + \cdots + k_j} [p_{\pi(\alpha)} - p_{\pi(\alpha)-1}],$$

which exists as a principal-value integral. Throughout we restrict $0 < t < 1$ and allow η to vary over compact subsets of $V_+^{\otimes(s-1)}$. (For the time variables, η^0 ranges over $\mathcal{R}_+^{s-1}$.) In order to show that the integral (A1) exists and to obtain simple estimates for its behavior near the boundary, we note the following trivial lemma.

Lemma 4: $\forall$ partitions $(k) = (k_1, k_2, \cdots, k_s)$ and permutations $\pi(k)$ there exist polynomials $Q_i(\eta^0, |\boldsymbol{\eta}|)$ which are strictly greater than zero when the η_j $1 \leq j \leq s-1$ vary over compact subsets of V_+, such that

$$\sum_{j=1}^{s-1} \tilde{P}_j^{\pi(k)} \eta_j \geq \sum_{\alpha \in I_{(k)}^\pi} \cosh \theta_\alpha\, Q_\alpha(\eta^0, |\boldsymbol{\eta}|) > 0.$$

Proof: The lemma follows in the case of an arbitrary partition (k) once it is proved for (1.5).

Consider the partial sums

$$\tilde{P}_j^\pi = \sum_{\alpha=1}^{j} [p_{\pi(\alpha)} - p_{\pi(\alpha)-1}],$$

$$\theta(\pm p_{\pi(\alpha)}^0) \quad \text{if} \quad \alpha \lesseqgtr \pi^{-1}[\pi(\alpha) + 1]$$

and suppose a given $p_{\pi(\alpha)}$ appears in $\tilde{P}_j^\pi$ for some $\pi|_{2n}|$ after cancellations. Then as $\tilde{P}_{2n}^\pi = 0$, $\exists \beta$: $j + 1 \leq \beta \leq 2n$ and $\pi(\beta) - 1 = \pi(\alpha)$ hence for this particular $\pi(\alpha)$; $p_{\pi(\alpha)}^0 > 0$. Similarly, we may show that a term $p_{\pi(\alpha)-1}$ remaining in $\tilde{P}_j^\pi$ after cancellation must have negative energy.

It is now clear that, after changing the integration variables appropriately in (A1), we may find a bound

$$\sum_{j=1}^{2n-1} \tilde{P}_j^\pi \eta_j \geq m\left(\sum_{i=1}^{2n} \cosh \theta_i\, Q_i(\eta^0, |\boldsymbol{\eta}|)\right) > 0,$$

where $Q(\eta^0, |\boldsymbol{\eta}|)$ is a polynomial made up of terms of the form $\eta_j^0 - |\boldsymbol{\eta}_j|$ with positive coefficients.

As a consequence, the exponential in (A1) with $0 < t < 1$ is in $\mathcal{S}$ for the variables θ_α, $\alpha \in I_{(k)}^\pi$ and (A1) exists. To study the limit $t \to O^+$, rewrite (A1) in the form

$$|G_s^{(k)}(\xi - it\eta)| \leq \frac{1}{(4\pi)^{|k|}} \sum_{\pi(k)} \left| \int \prod_{\alpha\; I_{(k)}^\pi} d\theta_\alpha H_{(k)}^\pi(\theta_\alpha) \exp\left(-t \sum_{j=1}^{s-1} \tilde{P}_j^{\pi(k)} \eta_j\right)\right|,$$

where $H_{(k)}^\pi(\theta)$ was defined by (2.5). From our remarks following (2.5) and (2.6), let us suppose that $h_{\alpha_i}(\theta)$ is in $\mathcal{S}$ and change variables to $x_j = \theta_{\alpha_j} - \theta_{\alpha_{j+1}}$, $1 \leq j \neq i \leq \lambda$, $x_i = \theta_{\alpha_i}$; whereupon the above integral becomes

$$|G_s^{(k)}(\xi - it\eta)| \leq \frac{1}{(4\pi)^{|k|}} \sum_{\pi(k)} \left| \int_{-\infty}^{\infty} dx_1 \cdots dx_n \prod_{\substack{j=1 \\ j \neq i}}^{\lambda} h_{\alpha_j}(x_j) h_{\alpha_i}\left[-\sum_{\substack{k=1 \\ k \neq i}}^{\lambda} x_k\right] \times \exp\left[-tm(\cosh x_i \tilde{\eta}^0 - \sinh x_i \tilde{\boldsymbol{\eta}})\right.\right. \tag{A2}$$

with $\tilde{\eta}$ a linear combination of the $\tilde{\eta}_j$ $1 \leq j \leq s-1$ as given by Lemma 4. Due to the convexity of the future light cone, $\tilde{\eta} \in V_+$ and varies over compact subsets with the η_j.

It is a straightforward exercise to show

$$\left| \int_{-\infty}^{\infty} dx_i \exp\left[-tm(\cosh x_i\tilde{\eta}^0 - \sinh x_i\tilde{\eta})\right] \right| \le K[\ln 1/t + K_0],$$

where K, K_0 are continuous on $0 < t < 1$, and uniformly bounded in $\tilde{\eta}$, t over their respective domains. The remaining integrals exist as convolutions of distributions in $0'_c$ with a function in $\mathcal{S}$. Further, using the fact that $h_{\alpha_j}(x)\, j \ne i$ is the Fourier transform of $(i \tanh \pi\sigma)^{l_i}$, and similarly $h_{\alpha_i}(x_i)$ of $(i \tanh \pi\sigma)^{l_i}/\cosh \pi\sigma$, we easily find

$$\left| \int_{-\infty}^{\infty} dx_1 \cdots \hat{d}x_i \cdots dx_\lambda h_{\alpha_1}(x_1) \cdots \times h_{\alpha_i}\left(-\sum_{\substack{j=1\\ j\ne i}}^{\lambda} x_j\right) \cdots h_{\alpha_\lambda}(x_\lambda) \right| \le (2\pi)^{\lambda-1}.$$

Applied to (A1) with λ bounded by $|k|$ gives the relation (3.6):

$$|G_s^{(k)}(\zeta)| \le \frac{(|k|-1)!}{2^{|k|}\pi} K_0[(\ln 1/t) + K_0'], \quad 0 < t < 1.$$

The second bound (3.11) requires a little more detail in order to give the power of t precisely. From (3.3),

$$\begin{aligned}
&\left| \int d\mathbf{x}_1 \cdots d\mathbf{x}_s f_1(\mathbf{x}_1) \cdots f_s(\mathbf{x}_s) \frac{\partial G_s^{(k)}(\zeta)}{\partial \zeta^0} \right| \\
&\le (m/\pi)^{|k|} \\
&\quad \times \sum_{\pi(k)} \left| \int \prod_{i=1}^{|k|} \frac{dp_i \delta \pm (p_i^2 - m^2)\tilde{P}_i^0}{[(p_1-p_2)^2(p_2-p_3)^2 \cdots (p_{|k|}-p_1)^2]^{\frac{1}{2}}} \right. \\
&\quad \times \exp\left[-i\sum_{j=1}^{s} \tilde{P}_j^0(\xi_j^0 - i\eta_j^0)\right] \prod_{j=1}^{s} \hat{f}_j(\mathbf{P}_j - \mathbf{P}_{j-1}) \\
&\quad \left. \times \exp\left(-\sum_{j=1}^{s-1} \tilde{\mathbf{P}}_j \boldsymbol{\eta}_j\right) \right|, \qquad \text{(A3)}
\end{aligned}$$

where there are at most $|k|$ terms in

$$\tilde{P}_j^0 = m \sum_{\alpha=1}^{k_1+\cdots+k_i} [\cosh \theta_{\pi(\alpha)} + \cosh \theta_{\pi(\alpha)-1}]$$

after changing the signs of the integration variables and omitting cancelled terms. As for Eq. (A2), by using Fourier transforms, (A3) is bounded by a similar integral in which $\hat{\phi}_t$ is replaced by $\tilde{\rho}_t$, where

$$\begin{aligned}
&\tilde{\rho}_t(\sigma_1, \cdots, \sigma_\lambda) \\
&= \frac{m}{(2\pi)^\lambda} \int_{-\infty}^{\infty} d\theta_1 \cdots d\theta_\lambda \cosh \theta_\beta \exp\left(-i\sum_{k=1}^{\lambda} \sigma_k \theta_k\right) \\
&\quad \times \exp\left(-i\sum_{j=1}^{s-1} \tilde{P}_j^{\pi(k)} \zeta_j^0 \prod_{j=1}^{s} [\tilde{f}_j(\mathbf{P}_j - \mathbf{P}_{j-1})]\right).
\end{aligned}$$

Replacing the test functions by $\|\hat{f}_j\|_{\epsilon_j}[1 + (\mathbf{P}_j^\pi - \mathbf{P}_{j-1}^\pi)^2]^{-\epsilon_j/2}$, $\epsilon_j > 0$, and changing variables to $u_k = t_k m \tilde{\eta}_k^0 \cosh \theta_k$, $1 \le k \le \lambda$, we see that $\prod_{i=1}^{\lambda}(1 + \sigma_i^2)\,|\tilde{\rho}_t|$ is bounded by 2^λ terms of the form

$$\begin{aligned}
&\prod_{j=1}^{s} \frac{\|\hat{f}_j\|_{\epsilon_j}}{(2\pi)^\lambda t_\beta \tilde{\eta}_\beta^0} \int_{mt_1\tilde{\eta}_1^0}^{\infty} \frac{du_1 t_1^{\mu_1} e^{-u_1}}{(u_1^2 - m^2 t_1^2 \tilde{\eta}_1^{02})^{\frac{1}{2}}} \cdots \\
&\times \int_{mt_\beta\tilde{\eta}_\beta^0} \frac{du_\beta t_\beta^{\mu_\beta} u_\beta e^{-u_\beta}}{(u_\beta^2 - m^2 t_\beta^2 \tilde{\eta}_\beta^{02})^{\frac{1}{2}}} \cdots \int_{mt_\lambda\tilde{\eta}_\lambda^0} \frac{du_\lambda t_\lambda^{\mu_\lambda} e^{-u_\lambda}}{(u_\lambda^2 - m^2 t_\lambda^2 \tilde{\eta}_\lambda^{02})^{\frac{1}{2}}} \\
&\times \{1 - P[u_k, (u_k^2 - t_k^2 m^2 \tilde{\eta}_k^{02})^{\frac{1}{2}}]\} \\
&\times \prod_{j=1}^{s} \left\{ t^2 + \left[\sum_\alpha \frac{(u_{\pi(\alpha)}^2 - t_{\pi(\alpha)}^2 m^2 \tilde{\eta}_{\pi(\alpha)}^{02})^{\frac{1}{2}}}{\tilde{\eta}_{\pi(\alpha)}^0} \right.\right. \\
&\quad \left.\left. \pm \frac{(u_{\pi(\alpha)-1}^2 - t_{\pi(\alpha)-1}^2 m^2 \tilde{\eta}_{\pi(\alpha)-1}^{02})^{\frac{1}{2}}}{\tilde{\eta}_{\pi(\alpha)-1}^0} \right]^2 \right\}^{-\epsilon_j/2}, \qquad \text{(A4)}
\end{aligned}$$

where $t = \min_{1 \le k \le \lambda} (t_k)$ and $P\,(a, b)$ is a polynomial. We have been a little free with the indices in this formula as the meaning is clear. It requires a simple argument to show that the polynomial P may be chosen so that the limit $t \to O^+$ is not affected. The numbers $\mu_k > 0$ represent the degree of divergence as $t_k \to O^+$ of our estimate for the test functions. We may always choose one or more of the ϵ_j such that there exists $0 < \epsilon < 1$ for which the singular part of (A4) as the t_k approach zero is bounded by

$$\begin{aligned}
&\prod_{j=1}^{s} \frac{\|\hat{f}_j\|_{\epsilon_j} t_\beta^{\epsilon-1}}{(2\pi)^\lambda \tilde{\eta}_\beta^0} \int_{t_1 m\tilde{\eta}_1^0}^{\infty} \frac{du_1 t^{\mu_1} e^{-u_1}}{u_1^{\mu} (u_1^2 - t_1^2 m^2 \tilde{\eta}_1^{02})^{\frac{1}{2}}} \cdots \\
&\times \int_{t_\beta m\tilde{\eta}_\beta^0}^{\infty} \frac{du_\beta u_\beta^{1-\epsilon} e^{-u_\beta}}{(u_\beta^2 - t_\beta^2 m^2 \tilde{\eta}_\beta^{02})^{\frac{1}{2}}} \cdots \int_{t_\lambda m\tilde{\eta}_\lambda^0} \frac{du_\lambda t_\lambda^{\mu_\lambda} e^{-u_\lambda}}{u_\lambda^{\mu_\lambda} (u_\lambda^2 - t_\lambda^2 m^2 \tilde{\eta}_\lambda^{02})^{\frac{1}{2}}}.
\end{aligned}$$

Letting $t = \min (t_k)$, this is bounded by

$$\frac{(\ln 1/t)^{\lambda-1}}{(2\pi)^\lambda} \prod_{j=1}^{s} \frac{\|\hat{f}_j\|_{\epsilon_j}}{t^{1-\epsilon}}.$$

Combining these estimates with the remarks following (A2), we find (3.11):

$$\left| \int d\mathbf{x}_1 \cdots d\mathbf{x}_s f_1(\mathbf{x}_1) \cdots f_s(\mathbf{x}_s) \frac{\partial G_s^{(k)}(\zeta)}{\partial \zeta_i^0} \right| \le \frac{M\,|k|!\, t^{\epsilon-1}}{\pi 2^{|k|}} [(\ln 1/t)^{|k|-1} + K_0'],$$

$0 < \epsilon < 1$, $0 < t < 1$, and n_j^0 $1 \le j \le s-1$ in compact subsets of $\mathcal{R}_+^{s-1}$.

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Canonical Realizations of the Galilei Group

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The general theory of the realizations of finite Lie groups by means of canonical transformations in classical mechanics, which has been developed in a preceding paper and already applied to the rotation group, is now applied to the Galilei group. Some complements to the general theory are introduced; in particular, a new kind of possible canonical realizations connected with the singularity surfaces of the functions $\mathfrak{Q}(y)$, $\mathfrak{P}(y)$, $\mathfrak{I}(y)$ are discussed (*singular realizations*). In agreement with the situation encountered in quantum mechanics, the constants $d_{\rho\sigma}$ appearing in the fundamental Poisson bracket relations among the infinitesimal generators ($\{y_\rho, y_\sigma\} = c^\tau_{\rho\sigma}y_\tau + d_{\rho\sigma}$) cannot all be reduced to zero. There remains a single independent constant m, which, in the physically significant cases ($m > 0$), represents the mass of the system. No physical interpretation seems to be attachable to the realizations corresponding to $m = 0$. For $m \neq 0$, two different kinds of *irreducible* realizations exist: one of a *singular* type which describes the free mass-point, and another of a *regular* type which describes a classical particle spin. A number of physical significant examples corresponding to *nonirreducible* realizations are thereafter discussed and the related *typical forms* are constructed: specifically, the cases dealt with are the rigid rod (linear rotator), the rigid body, and a system of two interacting mass points. It is shown that the problem of the construction of the variables of the *typical form* is equivalent to the determination of an appropriate solution of the time-independent Hamilton-Jacobi equation.

1. INTRODUCTION

In two preceding papers,[1] which we shall refer to as I and II, respectively, we developed a general theory of the realizations of finite-parameter Lie groups by means of the canonical transformations in classical mechanics and applied the theory to the rotation group. In this paper we deal with the more interesting case of the Galilei group. The paper has to be read in strict connection with II: here we use the same terminology and notations and assume the reader is familiar with all the results obtained there.

In order to clarify the physical meaning of the formal developments which follow and to expound a direct justification for our approach, we shall briefly outline some very general considerations about the problem of the invariance in the specific case of classical physics.

Let us synthetically denote, by the point P of an appropriate space, the set of variables which is sufficient to an observer O for the *complete description* of a physical system *at a certain instant of time.* Specifically, denoting by $\mathrm{P}(t)$ the position of P at the time t, we shall assume that $\mathrm{P}(t)$ can be expressed by means of a suitable transformation in terms only of t and $\mathrm{P}(0)$, and write

$$\mathrm{P}(t) = \mathsf{U}_t\mathrm{P}(0). \tag{1}$$

Let us consider, now, a second observer O' and assume that he can perform the same kind of measurements on the system that O does. Consequently, the time evolution of the system will be described by O' through a certain function $\mathrm{P}'(t')$ in the same space to which $\mathrm{P}(t)$ belongs. We assume also that the two trajectories $\mathrm{P}(t)$ and $\mathrm{P}'(t')$ are linked by a one-to-one correspondence. Clearly, such a correspondence can be expressed in particular also as a transformation connecting the point $\mathrm{P}(t)$, labeled by the time coordinate t with respect to O, with the point $\mathrm{P}'(t)$ labeled by the same time coordinate ($t' = t$) with respect to O'. This transformation will depend in general on the time coordinate itself so that we shall write

$$\mathrm{P}'(t) = \mathsf{L}_t\mathrm{P}(t). \tag{2}$$

We shall say that the two observers O and O' are equivalent if the time evolution of the system can be described by the two observers through the same transformation U_t, i.e., if one can also write

$$\mathrm{P}'(t) = \mathsf{U}_t\mathrm{P}'(0). \tag{3}$$

Substituting Eqs. (1) and (3) into Eq. (2), one obtains

$$\mathsf{U}_t\mathrm{P}'(0) = \mathsf{L}_t\mathsf{U}_t\mathrm{P}(0); \tag{4}$$

and using again Eq. (2), for $t = 0$, one has finally

$$\mathsf{U}_t\mathsf{L}_0 = \mathsf{L}_t\mathsf{U}_t, \tag{5}$$

or

$$\mathsf{L}_t = \mathsf{U}_t\mathsf{L}_0\mathsf{U}_t^{-1}. \tag{6}$$

Equation (6) expresses in the most general way the equivalence of the two observers O and O'.

[1] M. Pauri and G. M. Prosperi, J. Math. Phys. **7**, 366 (1966); **8**, 2256 (1967). For a general discussion of related matter see also: E. C. G. Sudarshan, "Principles of Classical Dynamics," lecture notes, Rochester (1963); D. G. Currie, T F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963).

Let us now consider a definite class of observers and assume that the set of all the space–time coordinate transformations connecting the observers form a group $\mathcal{G}$. We shall state that the physical laws are invariant with respect to the group $\mathcal{G}$ if the two following requirements are satisfied:

(1) Any two observers of the considered class are equivalent to each other in the sense specified above.

(2) The transformations L_t, relative to any pair of observers O and O' belonging to the class, depend only on the relation which connects O with O' and not on the particular choice of O. The second requirement directly implies that the set $\mathfrak{L}_t$ of the transformations L_t and, in particular, the set $\mathfrak{L}_0$ of the transformations L_0 will form a group homomorphic to $\mathcal{G}$.

Now, let us assume that the group $\mathcal{G}$ includes the time translation which connects two observers differing only in the choice of the origin of the time variable

$$t' = t - \tau \tag{7}$$

and denote by $\mathsf{E}(\tau)$ the transformation L_t corresponding to the operation (7). It holds in particular that

$$\mathrm{P}'(0) = \mathsf{E}_0(\tau)\mathrm{P}(0). \tag{8}$$

Since, obviously, in this case

$$\mathrm{P}'(0) = \mathrm{P}(\tau), \tag{9}$$

it follows that

$$\mathsf{E}_0(\tau) = \mathsf{U}_\tau; \tag{10}$$

that is, the time translation coincides with the transformation which expresses the time evolution of the system between the instants of time 0 and τ. Then Eq. (6) can be written as

$$\mathsf{L}_t = \mathsf{E}_0(t)\mathsf{L}_0\mathsf{E}_0^{-1}(t). \tag{11}$$

From Eqs. (10) and (11) one sees that the time-evolution law and the transformation properties of the *complete description* given by different observers at any given instant of time are completely determined once the transformations L_0 are given. Since the set $\mathfrak{L}_0$ provides a realization of the group $\mathcal{G}$, we conclude that, independently of time considerations, the problem of the construction of the most general theory invariant under $\mathcal{G}$ itself is reduced to that of constructing its possible realizations.

Here we are interested in the case of a system with a finite number of degrees of freedom and characterized by a set of canonical coordinates $q_1, \cdots, q_n$; $p_1, \cdots, p_n$, whose time evolution is described in Hamiltonian form. In this way, we are led to consider a situation in which the L_0's are canonical transformations and then to look for the possible canonical realizations $\mathfrak{R}$ of $\mathcal{G}$ which will be identified hereafter with the Galilei group. We stress once more that the realizations $\mathfrak{R}$, the construction of which is our main concern, do not directly involve time anyway, and that their elements physically represent the transformations which connect the canonical coordinates at time zero for equivalent observers. On the other hand, once the realization $\mathfrak{R}$ is given, the transformations defined by Eq. (11), which connect the canonical coordinates at any time t, provide a second realization $\mathfrak{R}_t$ of the group $\mathcal{G}$ by means of canonical transformations depending explicitly on time. On considering an element of $\mathcal{G}$, we shall refer to the corresponding elements of $\mathfrak{R}$ and $\mathfrak{R}_t$ as the *time-independent* and the *time-dependent images*, respectively.

In Sec. 2 we discuss some complements to the general theory given in I. More precisely, we treat in greater detail particular kinds of realizations which in I were only mentioned and are now of relevant interest for the Galilei group. Such realizations, which we call *singular realizations*, are distinguished by possessing invariant manifolds which lie on the boundary of the domain of the functions $\mathfrak{Q}(y)$, $\mathfrak{P}(y)$, $\mathfrak{I}(y)$ (see I).

In Sec. 3, after recalling some preliminaries on the Galilei group, the fundamental Poisson brackets among the canonical generators are established. The problem of the reduction of the constants $d_{\rho\sigma}$ to their minimum number is next discussed. As it is already known from the quantum case,[2] all the constants can be reduced to zero, apart from those relative to the Poisson brackets between the generators of the pure Galilean transformations and the homolog space translations, which remain equal to a single constant. Such a constant, which will be denoted by m, in the physical cases turns out to be the total mass of the system.

In Sec. 4, Scheme A (see II) is constructed, both in the case $m \neq 0$ and $m = 0$, and all the possible types of *singular realizations* are discussed. As in the quantum problem, only the realizations corresponding to $m > 0$ are directly significant from the physical point of view, while those corresponding to $m < 0$ are shown to be reducible to the former ones by means of an anticanonical transformation, and those corresponding to $m = 0$ are not directly interpretable as describing physical systems. In the case of $m > 0$, a direct physical meaning can be attached, in particular, to the *canonical invariants*, which, according to the most natural choice, can be taken for instance as the energy and the angular momentum in the center-of-mass system.

[2] J. M. Levy-Leblond, J. Math. Phys. 4, 776 (1963).

In Sec. 5 we construct the *irreducible* realizations, which correspond to fixed values of the *invariants*. Such *irreducible* realizations appear to be of particular interest. In the case $m > 0$, they are essentially of two kinds. The former one is of a singular type and corresponds to a system with three degrees of freedom and zero intrinsic angular momentum. The latter one is of a *regular* type and corresponds to a system with one more degree of freedom and a definite nonzero value of the intrinsic angular momentum. In both realizations, the Hamiltonian turns out to be of the form

$$H = \frac{\mathbf{p}^2}{2m} + \text{const},$$

so that they can be interpreted as corresponding to a free mass point and to a free particle with spin, respectively. This seems to be the most natural way for introducing the spin in classical mechanics. We stress that two constants characterize anyway the *irreducible* realizations: they are the values of m and of the intrinsic angular momentum. On the other hand, the value of the second *invariant* gives only the zero-point energy and does not affect the realizations. We are here in a case of *identical* realizations in the sense of the Sec. 4 of I. The construction of the *irreducible* realizations for the case $m = 0$ concludes the section.

In Secs. 6 and 7, the problem of constructing Scheme B for a number of physically significant examples corresponding to nonirreducible realizations is considered. The cases of the free rigid rod, the spherical and the symmetrical top, and a system of two mass points interacting through a Coulombian potential are explicitly solved. In the cases of the rotator and the spherical top, one of the two *invariants* is still fixed and gives the zero-point energy; consequently, the phenomenon of the *identical* realizations occurs again. In the case of the symmetrical top and of the system of two interacting mass points, the two *invariants* appear as canonical variables and there are no *inessential variables*. The same qualitative results hold for the case of the asymmetrical top: The corresponding realization turns out to be canonically equivalent to the realization corresponding to the symmetrical top.

Finally, in connection with the construction of Scheme B, it is noteworthy to notice that, with an appropriate choice of the *invariants*, the variables $Q(q,p)$ and $P(q,p)$ of the *typical form* are essentially Hamilton–Jacobi variables in the sense of the analytical mechanics. The problem of the construction of these variables can be consequently reduced to the finding of a suitable solution of the time-independent Hamilton–Jacobi equation. Conversely, the knowledge of the inverse canonical transformation amounts to obtaining the solution of the equations of motion. We take advantage of this remarkable circumstance for the actual construction of the variables of the *typical form* in the case of the two interacting mass points. In all other cases we shall follow directly the constructive procedure of the Theorem 2 proved in I.

2. COMPLEMENTS TO THE GENERAL THEORY: *SINGULAR* REALIZATIONS

As we pointed out in I and II, the canonical generators $y_1(q,p), \cdots, y_r(q,p)$ are not, in general, independent functions within a particular realization, because a certain number of relations of the form

$$f_\alpha(y_1, \cdots, y_r) = 0, \quad \alpha = 1, \cdots, s < r \tag{12}$$

may exist. Let us consider the form that such relations assume in terms of the expressions $\mathfrak{Q}$, $\mathfrak{P}$, and $\mathfrak{I}$ of Scheme A. When the y_ρ's are reexpressed in terms of the variables $\mathfrak{Q}$, $\mathfrak{P}$, and $\mathfrak{I}$, Eq. (12) becomes

$$g_\alpha(\mathfrak{Q}, \mathfrak{P}, \mathfrak{I}) = 0, \quad \alpha = 1, \cdots, s. \tag{13}$$

Then, using the fundamental relations

$$\begin{aligned} \{\mathfrak{Q}_i, \mathfrak{Q}_j\} &= \{\mathfrak{P}_i, \mathfrak{P}_j\} = \{\mathfrak{I}_t, \mathfrak{I}_{t'}\} \\ &= \{\mathfrak{Q}_i, \mathfrak{I}_t\} = \{\mathfrak{P}_j, \mathfrak{I}_t\} = 0, \\ \{\mathfrak{Q}_i, \mathfrak{P}_j\} &= \delta_{ij}; \quad i,j = 1, \cdots, h; \quad t, t' = 1, \cdots, k, \end{aligned} \tag{14}$$

it follows that

$$\frac{\partial g_\alpha}{\partial \mathfrak{Q}_i} = \frac{\partial g_\alpha}{\partial \mathfrak{P}_j} = 0, \quad \alpha = 1, \cdots, s. \tag{15}$$

This means that Eq. (13) does not imply constraints on the variables $\mathfrak{Q}_i$, $\mathfrak{P}_j$ so that Eqs. (12) are essentially relations among the *invariants* (cf. I, Sec. 3). However, we stress that in order to draw such a conclusion it is necessary that the manifold defined by Eqs. (12) lies inside the domain of the functions $\mathfrak{Q}(y)$, $\mathfrak{P}(y)$, $\mathfrak{I}(y)$. An exceptional situation occurs, instead, when the above manifold belongs to the singularity surfaces of the same functions. All the realizations corresponding to this last class will be called *exceptional* or *singular* realizations. In order to treat such *singular* realizations one has to proceed in the following way: Among the singularity manifold of the functions $\mathfrak{Q}(y)$, $\mathfrak{P}(y)$, $\mathfrak{I}(y)$ one must choose, in all possible ways, the submanifolds which are left invariant by the group transformations. Let us write

$$\begin{cases} h_1(y_1, \cdots, y_r) = 0, \\ \cdots \\ h_{r-u}(y_1, \cdots, y_r) = 0, \quad u < r, \end{cases} \tag{16}$$

as the equations for a given invariant submanifold. Then, from the expressions $y_1, \cdots, y_r$, one has to select a set $y'_1, \cdots, y'_u$ which is independent within the manifold (16). After that, one has to reapply the procedure given in I and to construct a system of new functions $\mathfrak{Q}'(y)$, $\mathfrak{P}'(y)$, $\mathfrak{I}'(y)$, i.e., according to the terminology introduced in II, a new Scheme A. We shall presently see, in particular (cf. Sec. 3), that the only unfaithful realizations occurring in the case of the Galilei group are just of the *singular* kind. On the other hand, we know that this is not true in general (see I, Sec. 4).

Another kind of singularity may occur in connection with the dependence of the functions of Scheme A on the constants $d_{\rho\sigma}$. Actually, it may happen that the original expressions of the variables $\mathfrak{Q}(y)$, $\mathfrak{P}(y)$, $\mathfrak{I}(y)$ lose their meaning for exceptional values of such constants. In this case Scheme A has to be independently constructed from the beginning, starting from these exceptional values of the $d_{\rho\sigma}$'s. Such a situation occurs in the case of the Galilei group in correspondence with the value $m = 0$.

3. GENERALITIES ON THE GALILEI GROUP

The general Galilei transformation will be written in the following way, adopting the passive point of view:

$$\mathbf{x}' = \mathrm{R}\mathbf{x} - \mathbf{v}t - \mathbf{a},$$
$$t' = t - \tau, \tag{17}$$

where R represents a pure rotation. The parameters of the group are then the three parameters $\boldsymbol{\omega}$ which characterize R, and $\mathbf{v}$, $\mathbf{a}$, and τ. The following special kinds of transformations are contained in Eqs. (17):

(a) $\mathbf{a} = \mathbf{v} = \tau = 0$, pure rotations,

(b) $\mathrm{R} = \mathrm{I}$, $\mathbf{v} = \tau = 0$, pure space translations,

(c) $\mathrm{R} = \mathrm{I}$, $\mathbf{a} = \tau = 0$, pure Galilean transformations or accelerations,

and finally

(d) $\mathrm{R} = \mathrm{I}$, $\mathbf{a} = \mathbf{v} = 0$, pure time translations.

The infinitesimal operators of the group relative to the transformations (a), (b), (c), and (d) will be denoted by $\mathcal{M}$, $\mathcal{T}$, $\mathcal{K}$, and $\mathcal{E}$, respectively. They have the form

$$\mathcal{M} = -\mathbf{x} \times \frac{\partial}{\partial \mathbf{x}}, \quad \mathcal{T} = -\frac{\partial}{\partial \mathbf{x}},$$
$$\mathcal{K} = -t\frac{\partial}{\partial \mathbf{x}}, \quad \mathcal{E} = -\frac{\partial}{\partial t}, \tag{18}$$

and satisfy the following commutation relations

$$\begin{aligned}
&[\mathcal{M}_i, \mathcal{M}_j] = \epsilon_{ijl}\mathcal{M}, \quad [\mathcal{T}_i, \mathcal{T}_j] = 0,\\
&[\mathcal{M}_i, \mathcal{T}_j] = \epsilon_{ijl}\mathcal{T}, \quad [\mathcal{K}_i, \mathcal{K}_j] = 0,\\
&[\mathcal{M}_i, \mathcal{K}_j] = \epsilon_{ijl}\mathcal{K}, \quad [\mathcal{T}_l, \mathcal{E}] = 0,\\
&[\mathcal{M}_l, \mathcal{E}] = 0, \quad [\mathcal{K}_l, \mathcal{E}] = \mathcal{T}_l\\
&[\mathcal{T}_i, \mathcal{K}_j] = 0, \quad (i, j, l = x, y, z).
\end{aligned} \tag{19}$$

Given a canonical realization $\mathfrak{R}$ of the group, the generators corresponding to $\mathcal{M}$, $\mathcal{T}$, $\mathcal{K}$, and $\mathcal{E}$ will be denoted by M, T, K, and E, respectively. According to the results established in I, the Poisson-bracket relations among such generators can be formally obtained from Eqs. (19), replacing the canonical generators for the group operators, the Poisson brackets for the commutators, and adding suitable constants $d_{\rho\sigma}$ to the right-hand side of the equations. The constants $d_{\rho\sigma}$ are not all independent but are linked by the system of conditions

$$\begin{aligned}
&d_{\rho\sigma} + d_{\sigma\rho} = 0 \quad (\rho, \sigma, \lambda = 1, \cdots, 10),\\
&c^{\tau}_{\rho\sigma}d_{\tau\lambda} + c^{\tau}_{\lambda\rho}d_{\tau\sigma} + c^{\tau}_{\sigma\lambda}d_{\tau\rho} = 0.
\end{aligned} \tag{20}$$

In the present case the restrictions implied by such conditions, with obvious meaning of the symbols, are

$$\begin{aligned}
&d_{M_iT_j} = d_{T_iM_j}, \quad d_{M_iK_j} = d_{K_iM_j},\\
&d_{T_iK_j} = 0 \quad (i \neq j),\\
&d_{T_iK_i} = d_{T_jK_j},\\
&d_{M_lT_l} = d_{M_lK_l} = d_{M_lE} = d_{T_iT_j}\\
&\qquad = d_{T_lE} = d_{K_iK_j} = 0,\\
&d_{M_iT_j} = \epsilon_{ijl}d_{K_lE}, \quad \text{for every } i \text{ and } j,\\
&d_{M_iM_j} = -d_{M_jM_i}, \quad d_{M_iT_j} = -d_{T_jM_i},\\
&d_{M_iK_j} = -d_{K_jM_i}.
\end{aligned} \tag{21}$$

Consequently, there are ten independent $d_{\rho\sigma}$ and the Poisson-bracket relations become

$$\begin{aligned}
&\{M_i, M_j\} = \epsilon_{ijl}M_l + d_{M_iM_j}, \quad \{T_i, T_j\} = 0,\\
&\{M_i, T_j\} = \epsilon_{ijl}T_l + d_{M_iT_j}, \quad \{K_i, K_j\} = 0,\\
&\{M_i, K_j\} = \epsilon_{ijl}K_l + d_{M_iK_j}, \quad \{M_l, E\} = 0,\\
&\{K_i, E\} = T_i + \tfrac{1}{2}\epsilon_{ijl}d_{M_jT_l}, \quad \{T_l, E\} = 0\\
&\{T_i, K_j\} = m\delta_{ij}, \quad (i, j = x, y, z).
\end{aligned} \tag{22}$$

In order to reduce the $d_{\rho\sigma}$'s to the minimum number, one has to consider the following equations (see I):

$$c^{\tau}_{\rho\sigma}\alpha_{\tau} = 0 \quad (\rho, \sigma, \tau = 1, \cdots, 10). \tag{23}$$

Since they give

$$\alpha_{M_i} = \alpha_{T_i} = \alpha_{K_i} = 0 \quad (i = x, y, z), \tag{24}$$

all the independent $d_{\rho\sigma}$'s but one can be put equal to zero by means of a suitable redefinition of the canonical generators $y_\rho \to y_\rho + \alpha_\rho$. Precisely performing the substitutions

$$\begin{aligned} M_i &\to M_i + \tfrac{1}{2}\epsilon_{ijl} d_{M_j M_l}, \\ T_i &\to T_i + \tfrac{1}{2}\epsilon_{ijl} d_{M_j T_l}, \\ K_i &\to K_i + \tfrac{1}{2}\epsilon_{ijl} d_{M_j K_l}, \end{aligned} \tag{25}$$

one obtains from Eqs. (22)

$$\begin{aligned} \{M_i, M_j\} &= \epsilon_{ijl} M_l, & \{T_i, T_j\} &= 0, \\ \{M_i, T_j\} &= \epsilon_{ijl} T_l, & \{K_i, K_j\} &= 0, \\ \{M_i, K_j\} &= \epsilon_{ijl} K_l, & \{M_l, E\} &= 0, \\ \{K_l, E\} &= T_l, & \{T_l, E\} &= 0, \\ \{T_i, K_j\} &= m\delta_{ij}, & (i, j &= x, y, z). \end{aligned} \tag{26}$$

The reason why the constant m cannot be reduced to zero is essentially that a transformation of the form

$$E \to E + \alpha_E \tag{27}$$

does not change the values of the $d_{\rho\sigma}$'s.

At this point, let us make a few remarks. First, recalling Eq. (10) and comparing the Hamilton equations and the structure of the canonical transformations corresponding to the infinitesimal time translation, one can see that the following relation holds:

$$E = -H \tag{28}$$

[cf. Eq. (10)], where H is the Hamiltonian of the system. Moreover, as to the determination of the number of the *invariants*, we observe that, as stated in I, such a number is directly provided by order r of the group minus the generic rank of the matrix $\|c^\tau_{\rho\sigma} y_\tau + d_{\rho\sigma}\|$. Using Eqs. (26), it can be easily seen that in our case this rank is eight, whatever be the value of m. Thus, the number of the *canonical invariants* for the Galilei group is two. Finally, before concluding this section, we want to say some more about the meaning of the constant m. We shall presently see that in the physically significant examples the constant m represents the total mass of the system and, consequently, m must be greater than zero. No physical meaning, it seems, directly attaches to the realizations for which $m = 0$. As for the realizations corresponding to $m < 0$, it can be shown that they are convertible into the positive mass cases by means of an anticanonical transformation. [A transformation $\bar{q} = \bar{q}(q, p)$, $\bar{p} = \bar{p}(q, p)$ will be called anticanonical if it satisfies the conditions $\{\bar{q}_i, \bar{q}_j\}_{qp} = \{\bar{p}_i, \bar{p}_j\}_{qp} = 0$, $\{\bar{q}_i, \bar{p}_i\}_{qp} = -\delta_{ij}$. For instance $\bar{q}_i = q_i, \bar{p}_j = -p_j$ $(i, j = 1, \cdots, n)$.] In order to see this, we note that

$$\{A, B\}_{\bar{q}\bar{p}} = -\{A, B\}_{qp}. \tag{29}$$

Because of the infinitesimal transformations of the original variables

$$\begin{aligned} q_i' &= q_i + \delta a^\rho \{y_\rho, q_i\}_{qp}, \\ p_j' &= p_j + \delta a^\rho \{y_\rho, p_j\}_{qp} \quad (i, j = 1, \cdots, n), \end{aligned} \tag{30}$$

it can be deduced that

$$\begin{aligned} \bar{q}_i' &= \bar{q}_i - \delta a^\rho \{y_\rho, \bar{q}_i\}_{\bar{p}\bar{q}}, \\ \bar{p}_j' &= \bar{p}_j - \delta a^\rho \{y_\rho, \bar{p}_j\}_{\bar{q}\bar{p}} \quad (i, j = 1, \cdots, n), \end{aligned} \tag{31}$$

so that the new generators can be expressed as

$$\bar{y}_\rho(\bar{q}, \bar{p}) = -y_\rho[q(\bar{q}, \bar{p}), p(\bar{q}, \bar{p})] \quad (\rho = 1, \cdots, r). \tag{32}$$

From Eqs. (29) and (32) one readily ascertains that the Poisson-bracket relations among the new generators, with respect to the new variables, are given again by Eqs. (26) except for the last one, which is changed in sign.

4. CONSTRUCTION OF SCHEME A

Let us consider first the case $m \neq 0$. Direct inspection of Eqs. (26) enables us to put

$$\begin{aligned} &\mathfrak{P}_1 = T_x, \quad \mathfrak{P}_2 = T_y, \quad \mathfrak{P}_3 = T_z, \\ &\mathfrak{Q}_1 = -\frac{K_x}{m}, \quad \mathfrak{Q}_2 = -\frac{K_y}{m}, \quad \mathfrak{Q}_3 = -\frac{K_z}{m}. \end{aligned} \tag{33}$$

Then we look for a function of the canonical generators, say $\Phi = \Phi(\mathbf{M}, \mathbf{T}, \mathbf{K}, E)$, which has zero Poisson brackets with all the expressions (33). This means that Φ has to satisfy the following system of partial differential equations:

$$\begin{aligned} \frac{\partial \Phi}{\partial M_y} K_z - \frac{\partial \Phi}{\partial M_z} K_y - m \frac{\partial \Phi}{\partial T_x} + \frac{\partial \Phi}{\partial E} T_x = 0, \quad \text{cycl;} \\ \frac{\partial \Phi}{\partial M_y} T_z - \frac{\partial \Phi}{\partial M_z} T_y + m \frac{\partial \Phi}{\partial K_x} = 0, \quad \text{cycl.} \end{aligned} \tag{34}$$

One can easily verify (see Appendix A) that a system of independent solutions is provided by the expressions

$$\mathbf{S} \equiv \mathbf{M} + \frac{\mathbf{K}}{m} \times \mathbf{T} \tag{35}$$

and

$$-W \equiv E + \frac{\mathbf{T}^2}{2m}, \tag{36}$$

which, moreover, satisfy the following Poisson-bracket relations:

$$\{S_i, S_j\} = \epsilon_{ijl} S_l, \tag{37}$$

$$\{S_l, W\} = 0 \quad (i, j, l = x, y, z). \tag{38}$$

Let us remark that Eqs. (37) are identical to the

Poisson-bracket relations obeyed by the canonical generators of the rotation group. Thus, finally, application of the results of II, Scheme A for the Galilei group can be summarized in the form:

$$\begin{array}{llllll} \mathfrak{P}_1 = T_x & \mathfrak{P}_2 = T_y & \mathfrak{P}_3 = T_z & \mathfrak{P}_4 = S_z & \mathfrak{J}_1 = \mathbf{S}^2 & \mathfrak{J}_2 = -W = E + \dfrac{\mathbf{T}^2}{2m} \\ \mathfrak{Q}_1 = -\dfrac{K_x}{m} & \mathfrak{Q}_2 = -\dfrac{K_y}{m} & \mathfrak{Q}_3 = -\dfrac{K_z}{m} & \mathfrak{Q}_4 = \arctan\dfrac{S_y}{S_x} & \cdots & \cdots . \end{array} \tag{39}$$

It will be apparent from the following that, within the physical realizations, $\mathfrak{Q}_1$, $\mathfrak{Q}_2$, $\mathfrak{Q}_3$ can be interpreted as the coordinates of the center of mass; $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$ as the components of the total linear momentum; S_1, S_2, S_3 as those of the intrinsic angular momentum; and $W = -\mathfrak{J}_2$ as the internal energy. Let us remark, in particular, that the variables appearing in the fourth and fifth columns represent Scheme A for the little group of $(E, \mathbf{T})$ (rotation group).

Now, we want to discuss the occurrence of *singular* realizations. The singularity manifold for the expressions of Scheme A is related to the function $\mathfrak{Q}_4 = \arctan S_y/S_x$, which is singular for

$$S_x + iS_y = 0 \tag{40}$$

and for

$$S_x - iS_y = 0. \tag{41}$$

(From the point of view of the functions of two complex variables, the function $\arctan z_1/z_2$ has the two singularity surfaces $z_1 + iz_2 = 0$, $z_1 - iz_2 = 0$. Such surfaces intersect at the point $z_1 = z_2 = 0$, which is consequently a point of singularity for the function.) However, this manifold is not an invariant one for the Galilei group. The only singular invariant submanifold belonging to the surface (40)–(41) is provided by

$$\mathbf{S} \equiv 0. \tag{42}$$

The Scheme A for the *singular* realizations corresponding to the condition (42) is obtained directly from the scheme (39) by suppressing the fourth and fifth columns. Thus, the scheme corresponds to unfaithful (trivial) realizations of the little group. We have

$$\begin{array}{llll} \mathfrak{P}_1 = T_x & \mathfrak{P}_2 = T_y & \mathfrak{P}_3 = T_z & \mathfrak{J} = -W = E + \dfrac{\mathbf{T}_2}{2m} \\ \mathfrak{Q}_1 = -\dfrac{K_x}{m} & \mathfrak{Q}_2 = -\dfrac{K_y}{m} & \mathfrak{Q}_3 = -\dfrac{K_z}{m} & \cdots . \end{array} \tag{39'}$$

Furthermore, if the matrix defined by Eq. (44) of I is constructed, one easily checks that, both in cases $\mathbf{S} \not\equiv 0$ and $\mathbf{S} \equiv 0$, the rank q_0 equals the order of the group. Consequently, no unfaithful realization of the particular kind studied in I, Sec. 4, exists for $m \neq 0$.

Next, let us consider the case $m = 0$. It is apparent that in this circumstance there is a number of variables of the scheme (39) which become singular. Therefore, according to the prescriptions given in Sec. 2, one has to refer back to Eqs. (26), substituting for the fifth-line Poisson brackets:

$$\{T_i, K_j\} = 0, \quad (i, j = x, y, z). \tag{43}$$

Scheme A, in this case, can be constructed in the form

$$\begin{array}{llllll} \mathfrak{P}_1 = M_z & \mathfrak{P}_2 = \mathbf{M}^2 & \mathfrak{P}_3 = \mathbf{M}\cdot\mathbf{T} & \mathfrak{P}_4 = E & \mathfrak{J}_1 = |\mathbf{K}\times\mathbf{T}|^2 & \mathfrak{J}_2 = \mathbf{T}^2 \\ \mathfrak{Q}_1 = \tan^{-1}\dfrac{M_y}{M_x} & \mathfrak{Q}_2 = \dfrac{1}{2M}\tan^{-1} \times \dfrac{M(\mathbf{M}\times\mathbf{T})_z}{(\mathbf{M}\cdot\mathbf{T})M_z - M^2T_z} & \mathfrak{Q}_3 = \dfrac{1}{T}\tan^{-1} \times \dfrac{T(\mathbf{K}\times\mathbf{T}\cdot\mathbf{M})}{(\mathbf{K}\times\mathbf{T})\cdot(\mathbf{M}\times\mathbf{T})} & \mathfrak{Q}_4 = \dfrac{\mathbf{K}\cdot\mathbf{T}}{T^2} & \cdots & \cdots \end{array} \tag{44}$$

where $M = |\mathbf{M}|$, $T = |\mathbf{T}|$. The detailed calculations are given in Appendix B.

The invariant singular manifolds related to the scheme (44) and the corresponding Scheme A can be summarized in the following table:

$$\mathbf{K}\times\mathbf{T}=0$$

$$\begin{array}{lllll} \mathfrak{P}_1 = M_z & \mathfrak{P}_2 = \mathbf{M}^2 & \mathfrak{P}_3 = E & \mathfrak{I}_1 = \mathbf{M}\cdot\mathbf{T} & \mathfrak{I}_2 = \mathbf{T}^2 \\ \mathfrak{Q}_1 = \arctan\dfrac{M_y}{M_x} & \mathfrak{Q}_2 = \dfrac{1}{2M}\arctan\dfrac{M(\mathbf{M}\times\mathbf{T})_z}{(\mathbf{M}\cdot\mathbf{T})M_z - M^2T_z} & \mathfrak{Q}_3 = \dfrac{\mathbf{K}\cdot\mathbf{T}}{T^2} & \cdots & \cdots \end{array} \qquad (44')$$

$$\mathbf{T}=0$$

$$\begin{array}{lllll} \mathfrak{P}_1 = M_z & \mathfrak{P}_2 = \mathbf{M}^2 & \mathfrak{I}_1 = \mathbf{M}\cdot\mathbf{K} & \mathfrak{I}_2 = \mathbf{K}^2 & \mathfrak{I}_3 = E \\ \mathfrak{Q}_1 = \arctan\dfrac{M_y}{M_x} & \mathfrak{Q}_2 = \dfrac{1}{2M}\arctan\dfrac{M(\mathbf{M}\times\mathbf{K})_z}{(\mathbf{M}\cdot\mathbf{K})M_z - M^2K_z} & \cdots & \cdots & \cdots \end{array} \qquad (44'')$$

$$\mathbf{T}=\mathbf{K}=0$$

$$\begin{array}{lll} \mathfrak{P}_1 = M_z & \mathfrak{I}_1 = \mathbf{M}^2 & \mathfrak{I}_2 = E \\ \mathfrak{Q}_1 = \arctan\dfrac{M_y}{M_x} & \cdots & \cdots \end{array} \qquad (44''')$$

$$\mathbf{M}=\mathbf{T}=\mathbf{K}=0$$

$$\mathfrak{I} = E \qquad (44'''')$$

The realizations of the types (44′), (44″), and (44‴) are the canonical analogs of the true unitary representations of the second, third, and fourth classes studied by Inönü and Wigner.[3] The realizations (44′) are faithful, while all the others are unfaithful. In particular, within the realizations (44″) the translation subgroup corresponds to the identity. If $\mathfrak{I}_3$ is not fixed, such realizations are faithful realizations of the homogeneous Galilei group plus time translations. The realizations (44‴) are faithful realizations of the rotation group plus time translations and those of the type (44⁗) are faithful realizations of the time translations group if, again, $\mathfrak{I}\equiv E$ is not fixed. We remark that in the cases (44″) and (44‴) the time translation E becomes a commutative subgroup. Consequently, if E reduces to a constant, a new cause of unfaithfulness arises (cf. I, Sec. 4) and the realizations become faithful realizations of the homogeneous Galilei group [isomorphic to the Euclidean group in three dimensions $E^+(3)$] and of the rotation group, respectively.

5. THE *IRREDUCIBLE* REALIZATIONS

The *irreducible* realizations can be directly constructed according to the following procedure: one starts from Scheme A and inverts the functions $\mathfrak{Q}(y)$, $\mathfrak{P}(y)$, $\mathfrak{I}(y)$, obtaining the expressions

$$y_\rho = y_\rho(\mathfrak{Q}, \mathfrak{P}, \mathfrak{I}). \qquad (45)$$

Then one introduces axiomatically a system of $2h$ canonical variables q, p and sets

$$\mathfrak{Q}_i = q_i, \quad \mathfrak{P}_j = p_j \quad (i,j = 1,\cdots,h). \qquad (46)$$

Finally, one prescribes certain definite values for the canonical *invariants* in their accessible domain, taking into account the results of I, Sec. 4.

In the case $m \neq 0$ one realizes from the discussion of the previous section that there are two kinds of *irreducible* realizations: *singular* realizations, for which

$$\mathbf{S}\equiv 0, \qquad (47)$$

corresponding to systems with three degrees of freedom, and *regular* realizations, for which $\mathbf{S}^2$ equals a positive constant s^2, corresponding to systems with four degrees of freedom. In the case $m = 0$, according to the above, there can be five different kinds of *irreducible* realizations, three of which are unfaithful; moreover, the realization corresponding to the scheme (44) is *regular*, while the other four are of a *singular* type.

A. *Irreducible Singular* Realizations for $m \neq 0$: the Free Mass Point

For the *singular* realizations, we have from (39′), (45), and (46)

$$\mathbf{T} = \mathbf{p},$$
$$\mathbf{K} = -m\mathbf{q}, \qquad (48)$$

so that Eqs. (35) and (47) give

$$\mathbf{M} = \mathbf{q}\times\mathbf{p}. \qquad (49)$$

[3] E. Inönü and E. P. Wigner, Nuovo Cimento **9**, 705 (1952).

Finally, from the actual structure of the invariant $\mathfrak{I}_2$, we deduce the Hamiltonian

$$H = -E = \frac{\mathbf{p}^2}{2m} + \text{const.} \tag{50}$$

Using Eqs. (48), (49), and (50), it is possible to write down explicitly the infinitesimal transformations of the various types:

(a) Pure rotations

$$\begin{cases} q_i' = q_i + \delta\omega_l \epsilon_{lik} q_k, \\ p_j' = p_j + \delta\omega_l \epsilon_{ljk} p_k; \end{cases} \tag{51'}$$

(b) Space translations

$$\begin{cases} q_i' = q_i - \delta a_i, \\ p_j' = p_j; \end{cases} \tag{51''}$$

(c) Pure Galilean transformations

$$\begin{cases} q_i' = q_i, \\ p_j' = p_j - m\delta v_j; \end{cases} \tag{51'''}$$

(d) Time translations

$$\begin{cases} q_i' = q_i + \dfrac{p_i}{m}\delta\tau, \\ p_j' = p_j, \quad (i, j = x, y, z). \end{cases} \tag{51''''}$$

An examination of the transformation laws (51), along with Eq. (50), shows that the realization corresponds to a free mass point with Cartesian coordinates q_x, q_y, q_z; momenta p_x, p_y, p_z; and mass m.

In particular, from Eqs. (51‴) we observe that it follows that, under the acceleration

$$\mathbf{x}' = \mathbf{x} - \mathbf{v}t, \tag{52}$$

the configurational variable $\mathbf{q}$ does not change, exactly as $\mathbf{x}$ does not for $t = 0$. This is in agreement with the fact that, as discussed in Sec. 1, the elements of the realization provide a connection among the canonical variables at time equal to zero. The transformation properties for the canonical variables at time t have to be constructed according to Eq. (6) as a product of a first time translation $(-t)$, the transformation at time equal zero, and a second time translation (t). Since the time translation commutes with the pure rotations, the space translations, and, obviously, other time translations, the transformation properties of $\mathbf{q}(t)$ and $\mathbf{p}(t)$, under such transformations, are directly obtained simply by replacing such expressions for $\mathbf{q}$ and $\mathbf{p}$ into Eqs. (51′), (51″), and (51⁗). As for the acceleration, Eq. (51‴) has to be replaced by

$$\begin{cases} q_i'(t) = q_i(t) - \delta v_i t, \\ p_j'(t) = p_j(t) - m\delta v_j, \end{cases} \tag{53}$$

in complete agreement with Eq. (52). Correspondingly, the generators of such infinitesimal transformations can be obtained from the time-independent ones by simply reexpressing the old variables $\mathbf{q}$ and $\mathbf{p}$ in terms of the new ones $\mathbf{q}(t)$, $\mathbf{p}(t)$. One obtains

$$\begin{aligned} \mathbf{T} &= \mathbf{p}(t), \\ \mathbf{K} &= -m\mathbf{q}(t) + \mathbf{p}(t) \cdot t, \\ \mathbf{M} &= \mathbf{q}(t) \times \mathbf{p}(t), \\ -E = H &= \frac{\mathbf{p}^2(t)}{2m} + \text{const.} \end{aligned} \tag{54}$$

Going back to the original discussion, we recall that Eqs. (50) and (54) provide the expression of the free mass-point Hamiltonian, if m is interpreted as the physical mass. In this connection, remembering what was said at the end of Sec. 3, we remark that if one performs the anticanonical transformation

$$\begin{aligned} \bar{q}_i &= q_i, \\ \bar{p}_j &= -p_j \quad (i, j = x, y, z), \end{aligned} \tag{55}$$

it follows that

$$\begin{aligned} \bar{\mathbf{T}} &= -\mathbf{T}(\bar{\mathbf{q}}, -\bar{\mathbf{p}}) = \bar{\mathbf{p}}, \\ \bar{\mathbf{K}} &= -\mathbf{K}(\bar{\mathbf{q}}, -\bar{\mathbf{p}}) = m\bar{\mathbf{q}}, \\ \bar{\mathbf{M}} &= -\mathbf{M}(\bar{\mathbf{q}}, -\bar{\mathbf{p}}) = \bar{\mathbf{q}} \times \bar{\mathbf{p}}, \\ \bar{E} &= -E(\bar{\mathbf{q}}, -\bar{\mathbf{p}}) = -\mathfrak{I}_2 + \frac{\bar{\mathbf{p}}^2}{2m}, \end{aligned} \tag{56}$$

so that, in the new variables, a new realization is defined in which the mass m and the "internal energy" $W \equiv -\mathfrak{I}_2$ are changed in sign.

In this case, we observe that the value of $-\mathfrak{I}_2 \equiv W$ gives merely the zero-point energy. The physical arbitrariness of this quantity is reflected in the fact that the canonical realizations corresponding to different values of $\mathfrak{I}_2$ are *identical* ones. This case is just an example of the more general situation discussed in Sec. 4 of I [see also what was said in Sec. 2 of this paper in connection with Eq. (27)].

B. *Irreducible Regular* Realizations for $m \neq 0$: the Particle with Spin

We want to consider now the general *irreducible* realizations of a *regular* type. In this case, $\mathfrak{I}_1 \equiv \mathbf{S}^2$ has to equal identically a positive constant s^2, and $\mathfrak{I}_2$ can assume any real constant value as before. Such *regular* realizations can be explicitly constructed by introducing two additional canonical variables. Indeed, from the scheme (39) and Eq. (46), using now the variables χ and p_χ for q_4 and p_4, one has

$$\begin{aligned} \mathbf{T} &= \mathbf{p}, \\ \mathbf{K} &= -m\mathbf{q}, \end{aligned} \tag{57}$$

and

$$\begin{cases} S_x = (s^2 - p_\chi^2)^{\frac{1}{2}} \cos \chi\,, \\ S_y = (s^2 - p_\chi^2)^{\frac{1}{2}} \sin \chi\,, \\ S_z = p_\chi\,, \quad \mathbf{S}^2 = s^2. \end{cases} \tag{58}$$

Consequently, one also has [see Eq. (35)]

$$\mathbf{M} = \mathbf{q} \times \mathbf{p} + \mathbf{S}, \tag{59}$$

and, finally,

$$H = -E = \frac{\mathbf{p}^2}{2m} + \text{const.} \tag{60}$$

Such a realization can be plainly interpreted as corresponding to a free particle with spin s (see II, Secs. 4, 5 where a discussion of the connections with spinor theory is also given). The same remarks as before hold true about the zero-point energy.

C. The *Irreducible* Realizations for $m = 0$

The *irreducible* realizations for $m = 0$ should be obtained from Eqs. (44), (44′), (44″), (44‴), and (44⁗) following the procedure already described. We shall not do this in detail, but we want to make a few remarks.

The realizations corresponding to Schemes A (44″), (44‴), and (44⁗), as we have already said, are unfaithful realizations, i.e., realizations of factor groups and, consequently, they are not interesting for the Galilei group directly. The realizations corresponding to the schemes (44) and (44′) are instead faithful ones and stand, in some way, in analogy with the *regular* and the *singular* realizations for $m \neq 0$. Actually, they can be viewed as limiting cases of the previous situations when the mass goes to zero and consequently the velocity and the center-of-mass coordinates go to infinity. In this connection, we can see that there is no possible way to introduce into the framework of such realizations center-of-mass coordinates and linear-momentum variables satisfying reasonable physical requirements. As to the center-of-mass coordinates, obvious requirements are that under infinitesimal rotations, space translations, and accelerations, they shall transform respectively as

$$\begin{aligned} Q'_j &= Q_j + \delta\omega_l \epsilon_{ljk} Q_k, \\ Q'_j &= Q_j - \delta a_j, \\ Q'_j &= Q_j, \quad (j, l, k = x, y, z). \end{aligned} \tag{61}$$

Consequently, Q_j has to satisfy the following relations:

$$\begin{aligned} \{M_i, Q_j\} &= \epsilon_{ijk} Q_k, \\ \{T_i, Q_j\} &= -\delta_{ij}, \\ \{K_i, Q_j\} &= 0 \quad (i, j, k = x, y, z). \end{aligned} \tag{62}$$

It is readily seen that a quantity obeying such equations cannot have zero Poisson brackets with the canonical *invariants* appearing in the schemes (44) and (44′) and so it cannot be a function of the infinitesimal generators alone. As for the linear momentum, we should assume the following transformation properties under rotations, space translations, and accelerations, respectively:

$$\begin{aligned} P'_j &= P_j + \delta\omega_l \epsilon_{ljk} P, \\ P'_j &= P_j, \\ P'_j &= P_j - \mu \delta v_j \quad (j, l, k = x, y, z), \end{aligned} \tag{63}$$

where μ should be the mass of the system, so that

$$\begin{aligned} \{M_i, P_j\} &= \epsilon_{ijk} P_k, \\ \{T_i, P_j\} &= 0, \\ \{K_i, P_j\} &= -\mu \delta_{ij} \quad (i, j, k = x, y, z). \end{aligned} \tag{64}$$

Now, Eqs. (64) are incompatible with the schemes (44) and (44′) unless $\mu = 0$. This is straightforward, as before, in the case of scheme (44). As for scheme (44′), from the first Eq. (64), we must have

$$\{\mathbf{M} \cdot \mathbf{T}, P_j\} = -(\mathbf{T} \times \mathbf{P})_j = 0 \quad (j = x, y, z), \tag{65}$$

so that

$$P_j = \lambda T_j \quad [\text{with } \{M_l, \lambda\} = 0 \ (l = x, y, z)]. \tag{66}$$

Moreover, in order to satisfy the remaining Eqs. (64), it must be that

$$\{T_i, \lambda\} = 0, \quad \{K_i, \lambda\} T_j = -\mu \delta_{ij}. \tag{67}$$

Taking $j \neq i$, the second equation gives, for any i,

$$\{K_i, \lambda\} = 0 \tag{68}$$

and, consequently,

$$\{K_i, P_j\} = 0. \tag{69}$$

In both cases (44) and (44′), if also we require that P_j be a constant of the motion $\{E, P_j\} = 0$, we must have Eq. (66) with $\lambda = \text{const}$. However, such a momentum variable does not correspond to a zero-mass system.

6. *NONIRREDUCIBLE* REALIZATIONS: FREE RIGID SYSTEMS

We want to discuss, now, some simple classic examples of mechanical systems corresponding to *nonirreducible* realizations. In analogy with what was done in the case of the rotation group, our attitude will be to consider the transformation properties of the canonical coordinates as already known on the basis of their physical meaning; then to deduce the explicit expressions of the generators of the infinitesimal transformations; finally to construct the corresponding Scheme B. In this section we shall discuss free rigid systems: precisely, the rigid rod, the spherical top, the symmetrical and the asymmetrical top.

A. Free Rigid Rod

We intend for a rigid rod or linear rotator, in analogy with II, a system characterized by the coordinates of a mass center q_x, q_y, q_z, by two angular coordinates φ, θ specifying an orientation, and by the conjugate momenta $p_x, p_y, p_z; p_\varphi, p_\theta$.

Under a pure rotation, the mass-center coordinates and the conjugate momenta transform as vectors, while the transformation properties of the angular variables φ, θ and of p_φ, p_θ are those of the same coordinates of the linear rotator [cf. II (Sec. 3, ii)]. Under a space translation all the variables remain unchanged except for the mass-center coordinates which transform according to the law $\mathbf{q}' = \mathbf{q} - \delta\mathbf{a}$. Under an acceleration, in the *time-independent image*, only the total momentum $\mathbf{p}$ transforms, according to the law $\mathbf{p}' = \mathbf{p} - m\delta\mathbf{v}$. Finally, the transformation properties under a time translation are obviously determined by the structure of the Hamiltonian. Summing up, we can write the following expressions for the infinitesimal generators [see II (Sec. 3, ii)]:

$$\begin{aligned} \mathbf{T} &= \mathbf{p}, \\ \mathbf{K} &= -m\mathbf{q}, \\ \mathbf{M} &= \mathbf{q} \times \mathbf{p} + \mathbf{S}, \\ -E = H &= \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{S}^2}{2I} + \text{const} \\ &= \frac{\mathbf{p}^2}{2m} + \frac{1}{2I}\left(p_\theta^2 + \frac{1}{\sin^2\theta} p_\varphi^2\right) + \text{const}, \end{aligned} \tag{70}$$

with

$$\begin{cases} S_x = -\sin\varphi p_\theta - \cot\theta \cos\varphi p_\varphi, \\ S_y = \cos\varphi p_\theta - \cot\theta \sin\varphi p_\varphi, \\ S_z = p_\varphi, \quad \mathbf{S}^2 = p_\theta^2 + \dfrac{1}{\sin^2\theta} p_\varphi^2, \end{cases} \tag{70$'$}$$

where m and I are the mass and the moment of inertia of the rod. The angular momentum in the center-of-mass system is denoted by $\mathbf{S}$.

In constructing Scheme B, the only variables which are not trivially obtained from Eqs. (70) and (70′) are Q_5 and P_5. Using the results of II (Sec. 3, ii and Appendix A), one sees that a possible choice is provided by

$$\begin{aligned} P_5 &\equiv \mathfrak{I}_1 = p_\theta^2 + \frac{1}{\sin^2\theta} p_\varphi^2, \\ Q_5 &= \frac{1}{2S} \arctan \frac{p_\theta \tan\theta}{S}, \quad S = |\mathbf{S}|. \end{aligned} \tag{71}$$

Moreover, since the variables are ten in number, one of the *invariants* must be fixed. However, one directly ascertains from Eqs. (70) that this cannot be the case for $\mathfrak{I}_2$ itself. A correct choice for the second invariant in the Scheme B is instead

$$\mathfrak{I}_2' \equiv \mathfrak{I}_2 + \frac{1}{2I}\mathfrak{I}_1 \quad \text{[see I and II (Sec. 1)]}, \tag{72}$$

which, again, gives the physically irrelevant zero-point energy, with the same consequences as before. The physical meaning of the variable Q_5 as a suitable angle of rotation has been illustrated in II [Sec. 3, ii], where its role was played by the variable Q_2. The Scheme B for this case together with the preceding and the following ones is summarized at the end of the paper (see Table I).

B. Free Rigid Body

The canonical coordinates in this case can be chosen to be $q_x, q_y, q_z; \varphi, \theta, \psi$ and $p_x, p_y, p_z; p_\varphi, p_\theta, p_\psi$; as before, the q_x, q_y, q_z, are the center-of-mass coordinates and φ, θ, ψ the Euler angles specifying the body orientation. (The conventions used in Ref. 4 are adopted throughout.) Proceeding as in the previous case and using the results of II (Sec. 3, iii), the canonical generators can be written as

$$\begin{aligned} \mathbf{T} &= \mathbf{p}, \\ \mathbf{K} &= -m\mathbf{q}, \\ \mathbf{M} &= \mathbf{q} \times \mathbf{p} + \mathbf{S}, \\ -E = H &= \frac{\mathbf{p}^2}{2m} + \frac{\Sigma_\xi^2}{2I_1} + \frac{\Sigma_\eta^2}{2I_2} + \frac{\Sigma_\zeta^2}{2I_3} + \text{const}, \end{aligned} \tag{73}$$

where

$$\begin{cases} S_x = \cos\varphi p_\theta + \dfrac{\sin\varphi}{\sin\theta} p_\psi - \cot\theta \sin\varphi p_\varphi, \\ S_y = \sin\varphi p_\theta - \dfrac{\cos\varphi}{\sin\theta} p_\psi + \cot\theta\cos\varphi p_\varphi, \\ S_z = p_\varphi, \end{cases} \tag{73$'$}$$

$$\mathbf{S}^2 = p_\theta^2 + \frac{1}{\sin^2\theta}(p_\varphi^2 + p_\psi^2) - \frac{2\cot\theta}{\sin\theta} p_\varphi p_\psi,$$

where I_1, I_2, I_3 are the principal moments of inertia and Σ_ξ, Σ_η, Σ_ζ are the expressions obtained from S_x, S_y, S_z interchanging φ and ψ, p_φ and p_ψ; they represent the three components of the intrinsic angular momentum $\mathbf{S}$ in the body system which we assume here to be defined by the principal axes of inertia [cf. II, (Sec. 3, iii)]. We recall also that

$$\{S_i, \Sigma_j\} = 0 \quad (i = x, y, z; j = \xi, \eta, \zeta), \tag{74}$$

[4] H. Goldstein, *Classical Mechanics* (Addison-Wesley Publishing Co., Reading, Mass., 1959), Chap. 4.

and obviously

$$\{\Sigma_i, \Sigma_j\} = \epsilon_{ijk}\Sigma_k \quad (i, j, k = \xi, \eta, \zeta). \tag{75}$$

In order to construct Scheme B we need to discuss different cases separately:

Spherical Top: $I_1 = I_2 = I_3 \equiv I$

The Hamiltonian becomes

$$H = -E = \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{S}^2}{2I} + \text{const.} \tag{76}$$

Thus one sees that, as in the case of the linear rotator,

$$\mathfrak{I}_2' \equiv \mathfrak{I}_2 + \frac{1}{2I}\mathfrak{I}_1 = \text{const} \tag{77}$$

is true. Then, using the results obtained in II (Sec. 3, iii and Appendix D), Scheme B can be completed by setting

$$\begin{aligned} P_5 &\equiv \mathfrak{I}_1 = p_\theta^2 + \frac{1}{\sin^2\theta}(p_\varphi^2 + p_\psi^2) - \frac{2\cot\theta}{\sin\theta} p_\varphi p_\psi, \\ Q_5 &= \frac{1}{2S}\arctan \frac{p_\theta \tan\theta}{S - \dfrac{p_\varphi p_\psi}{S\cos\theta}}, \\ P_6 &= \Sigma_\zeta, \\ Q_6 &= \arctan\frac{\Sigma_\eta}{\Sigma_\xi}. \end{aligned} \tag{78}$$

In conclusion, there are two variables P_5 and Q_5 in the second set [cf. I (Sec. 3)], one fixed *invariant* $\mathfrak{I}_2'$ and two *inessential* variables P_6, Q_6 in the fourth set. We notice in particular that, for a time translation,

$$Q_5' = Q_5 + \frac{\tau}{2I} \tag{79}$$

is true in agreement with the fact that, according to the discussion given in II (Sec. 3, iii), the expression $2SQ_5$ represents the angle of rotation around the intrinsic angular momentum **S**—more precisely, the angle between the half-planes from **S** to the z and ζ axes, respectively.

Symmetrical Top: $I_1 = I_2 \equiv I \neq I_3$

The Hamiltonian can be written

$$\begin{aligned} H = -E &= \frac{\mathbf{p}^2}{2m} + \frac{\Sigma_\xi^2 + \Sigma_\eta^2}{2I} + \frac{\Sigma_\zeta^2}{2I_3} + \text{const} \\ &= \frac{\mathbf{p}^2}{2m} + \frac{\mathbf{S}^2}{2I} + \frac{I - I_3}{2II_3} p_\psi^2 + \text{const.} \end{aligned} \tag{80}$$

No function of the *invariants* can be constructed in this case, which is identically equal to a constant; consequently, the fourth set is empty. The *invariant* $\mathfrak{I}_2'$ defined above becomes

$$\mathfrak{I}_2' = \frac{I_3 - I}{2I_3 I} p_\psi^2 + \text{const,} \tag{81}$$

so that it has zero Poisson bracket with the expression of Q_5 already introduced for the spherical top. Thus, we can set

$$\begin{aligned} P_5 &\equiv \mathfrak{I}_1 = p_\theta^2 + \frac{1}{\sin^2\theta}(p_\varphi^2 + p_\psi^2) - \frac{2\cot\theta}{\sin\theta} p_\varphi p_\psi, \\ Q_5 &= \frac{1}{2S}\arctan \frac{p_\theta \tan\theta}{S - \dfrac{p_\varphi p_\psi}{S\cos\theta}}, \\ P_6 &\equiv \mathfrak{I}_2' = \frac{I_3 - I}{2I_3 I} p_\psi^2. \end{aligned} \tag{82}$$

The variable Q_6 has to be, for instance, a function of the quantities Σ_ζ, $\arctan \Sigma_\eta/\Sigma_\xi$ such that

$$\{Q_6, P_6\} = 1. \tag{83}$$

One readily finds

$$Q_6 = \frac{I_3 I}{I_3 - I}\frac{1}{\Sigma_\zeta}\arctan\frac{\Sigma_\eta}{\Sigma_\xi}. \tag{84}$$

Under a time translation, Q_5 and Q_6 transform in the following way:

$$\begin{aligned} Q_5' &= Q_5 + \frac{\tau}{2I}, \\ Q_6' &= Q_6 - \tau. \end{aligned} \tag{85}$$

As to their physical meaning, we assume that the ζ axis coincides with the axis of the symmetrical top; then the expression $2SQ_5$ provides, as before, the angle between the two half-planes from **S** to the z and the ζ axes, respectively, i.e., in this case, the precession angle. The quantity $(I - I_3/II_3)\, \Sigma_\zeta\, Q_6$ gives, instead, the angle between the half-plane (S–ζ) and (ξ–ζ), i.e., essentially, the proper rotation angle of the body. Let us remark that a possible different choice for the variables of the second set, precisely that which retains the original form of the invariants $\mathfrak{I}_1$, $\mathfrak{I}_2$, is accomplished by putting

$$\begin{aligned} \bar{P}_5 &= P_5 \equiv \mathfrak{I}_1, \\ \bar{Q}_5 &= Q_5 + \frac{1}{2I} Q_6, \\ \bar{P}_6 &= P_6 - \frac{1}{2I} P_5 \equiv \mathfrak{I}_2, \\ \bar{Q}_6 &= Q_6. \end{aligned} \tag{86}$$

In this case $\bar{Q}_5$ does not change under the time translation, i.e., it is a constant of motion.

Asymmetrical Top: $I_1 \neq I_2$, $I_2 \neq I_3$, $I_1 \neq I_3$

In this case also there is no variable in the fourth set. It is interesting to remark that the canonical realization is canonically equivalent to the previous one of the symmetrical top, as it can be easily seen. For instance, if one puts

$$P_5 = \mathfrak{J}_1, \quad P_6 = \mathfrak{J}_2, \tag{87}$$

all the canonical generators, in terms of the variables of Scheme B, have the same form in the two cases. In particular, the Hamiltonian takes the form

$$H = \frac{\mathbf{p}^2}{2m} - P_6 \tag{88}$$

for the asymmetrical top and [see Eqs. (86)]

$$H = \frac{\mathbf{p}^2}{2m} - \bar{P}_6 \tag{89}$$

for the symmetrical top. On the contrary, the realizations for the symmetrical top and the asymmetrical top are not canonically equivalent to the realization corresponding to the spherical top. In addition, let us stress that, since the differences between such realizations are of a strictly dynamical character, they do not appear in the corresponding realizations of the rotation group, as it is apparent from II (see also footnote to Table I).

The actual construction of the variables Q_5 and Q_6 for the asymmetrical top involves the solution of a very complicated system of partial differential equations leading, as could be expected, to elliptic functions. Though the system is solvable, we shall not quote here the results and no further detail will be added about this point. Of course, the physical meaning of the variables Q_5 and Q_6 is no longer a simple one: indeed, their transformation properties under time translations remain very simple but we know that in this case there is no significant angle varying linearly with time.

7. *NONIRREDUCIBLE* REALIZATIONS; SYSTEM OF MASS POINTS

Let us consider n mass points and denote with $\mathbf{q}_i, \mathbf{p}_i$ the canonical variables of the ith mass point and with m_i its mass; then consider the product of the realizations corresponding to each mass point. In this case

$$\begin{aligned} \mathbf{T} &= \sum_i \mathbf{p}_i, \\ \mathbf{K} &= -\sum_i m_i \mathbf{q}_i, \\ \mathbf{M} &= \sum_i \mathbf{q}_i \times \mathbf{p}_i, \\ -E = H &= \sum_i \frac{\mathbf{p}_i^2}{2m_i} \quad (i = 1, \cdots, n). \end{aligned} \tag{90}$$

As for the constant m appearing in Eqs. (22), one has

$$m = \sum_i m_i \quad (i = 1, \cdots, n). \tag{91}$$

The quantities

$$\begin{aligned} \mathbf{Q} &= -\frac{\mathbf{K}}{m} = \frac{1}{m} \sum_i m_i \mathbf{q}_i, \\ \mathbf{P} &= \mathbf{T} = \sum_i \mathbf{p}_i, \\ \mathbf{S} &= \sum_i (\mathbf{q}_i - \mathbf{Q}) \times \mathbf{p}_i, \\ -\mathfrak{J}_2 = W &= H - \frac{\mathbf{P}^2}{2m} = \sum_i \frac{1}{2m_i}\left(\mathbf{p}_i - \frac{m_i}{m}\mathbf{P}\right)^2 \\ &\qquad (i = 1, \cdots, n) \end{aligned} \tag{92}$$

are the coordinates of the center-of-mass, the total linear momentum, the angular momentum, and the energy taken in the center-of-mass frame. The realization corresponds clearly to a system of free points.

In order to introduce an interaction, one can proceed in the following way: The expressions for $\mathbf{T}$, $\mathbf{K}$, and $\mathbf{M}$ are left unchanged, and the Hamiltonian is modified as

$$H = -E = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + U(\mathbf{q}_l, \mathbf{p}_j) \quad (i, l, j = 1, \cdots, n). \tag{93}$$

The requirement that the Poisson-bracket relations (22) shall be still satisfied leads to

$$\{M_l, U\} = \{T_l, U\} = \{K_l, U\} = 0 \quad (l = x, y, z). \tag{94}$$

Relations (94) imply that U must depend only on the scalar products built up from coordinates $\mathbf{q}_i - \mathbf{Q}$ and momenta $\mathbf{p}_i - (m_i/m)\mathbf{P}$ (not all independent) in the center-of-mass frame or, which is the same, from relative coordinates $\mathbf{q}_i - \mathbf{q}_j$ and relative velocities $\mathbf{p}_i/m_i - \mathbf{p}_j/m_j$. We remark that we could have modified the expression of $\mathbf{K}$ as well, by setting

$$\mathbf{K} = -\sum_i m_i \mathbf{q}_i + \mathbf{V}(\mathbf{q}_l, \mathbf{p}_j) \quad (i, l, j = 1, \cdots, n). \tag{95}$$

However, with the aid of arguments parallel to those given in the quantum case,[5] it can be shown that $\mathbf{V}(\mathbf{q}_i, \mathbf{p}_j)$ can be put equal zero without loss of generality.

Here, we shall limit ourselves to discuss in detail the case of two mass points. Let us introduce the relative coordinate $\boldsymbol{\xi} = \mathbf{q}_1 - \mathbf{q}_2$ and the conjugate

[5] L. L. Foldy, Phys. Rev. **122**, 275 (1961); D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Rev. Mod. Phys. **35**, 350 (1963), p. 367.

momentum

$$\mathbf{p}_\xi = \mu\dot{\boldsymbol{\xi}} = \frac{m_2}{m}\mathbf{p}_1 - \frac{m_1}{m}\mathbf{p}_2,$$

where $\mu = m_1 m_2/m$ is the reduced mass and now, obviously, $m = m_1 + m_2$. The angular momentum in the center-of-mass frame and the internal energy can be written as

$$\mathbf{S} = \boldsymbol{\xi} \times \mathbf{p}_\xi, \tag{96}$$

$$W \equiv -\mathfrak{I}_2 = \frac{1}{2\mu}\mathbf{p}_\xi^2 + U(|\boldsymbol{\xi}|, |\mathbf{p}_\xi|, \boldsymbol{\xi}\cdot\mathbf{p}_\xi). \tag{97}$$

If polar coordinates r, φ, θ, rather than Cartesian ones, are used to specify $\boldsymbol{\xi}$, and the corresponding conjugate momenta p_r, p_φ, p_θ are introduced, one obtains

$$\begin{cases} S_x = -\sin\varphi p_\theta - \cot\theta\cos\varphi p_\varphi, \\ S_y = \cos\varphi p_\theta - \cot\theta\sin\varphi p_\varphi, \\ S_z = p_\varphi, \quad \mathbf{S}^2 = p_\theta^2 + \dfrac{1}{\sin^2\theta}p_\varphi^2, \end{cases} \tag{98}$$

$$\begin{aligned} W \equiv -\mathfrak{I}_2 &= \frac{1}{2\mu}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\varphi^2\right) + U \\ &= \frac{p_r^2}{2\mu} + \frac{\mathbf{S}^2}{2\mu r^2} + U. \end{aligned} \tag{99}$$

Note that the expressions of the components of $\mathbf{S}$ coincide with the corresponding expressions for the intrinsic angular momentum of the linear rotator.

We see from Eqs. (98) and (99) that none of the *invariants* reduce to a constant. Consequently, we can put $P_5 \equiv \mathfrak{I}_1 = \mathbf{S}^2$ and $P_6 \equiv \mathfrak{I}_2 = -W$. In order to construct the variables Q_5 and Q_6 of Scheme B, we could apply the same procedure used in the previous examples, which is directly based on the Theorem 2 of I. We do this in Appendix C for the case $U \equiv 0$. Here, instead, we want to have the opportunity of introducing and illustrating a different approach which displays general interest, since it shows the connections between our group-theoretic formulation and the classical formalism of Hamilton–Jacobi and, in addition, it appears directly profitable for actual calculations (for instance) in the case of purely positional central interactions.

We observe that, in the case of the Galilei group, the variables $Q(q,p)$ and $P(q,p)$ of the *typical form* are essentially Hamilton–Jacobi variables in the sense of the analytical mechanics. Therefore, *the problem of the construction of Scheme B is essentially reduced to finding a suitable solution of the time-independent Hamilton–Jacobi equation. Conversely, the knowledge of Scheme B provides the solution of the equations of motion.*

If we assume that we have $U = U(r)$, and we separate the center-of-mass variables, the time-independent Hamilton–Jacobi equation can be written in the form

$$\frac{1}{2\mu}\left[\left(\frac{\partial \mathcal{S}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2 + \frac{1}{r^2\sin\theta}\left(\frac{\partial \mathcal{S}}{\partial \varphi}\right)^2\right] = w. \tag{100}$$

A complete integral of this equation is provided by (see for instance Ref. 6)

$$\begin{aligned} \mathcal{S} = \alpha_\varphi\varphi &+ \int_{\theta_0}^{\theta} d\theta'\left\{\alpha_\theta^2 - \frac{\alpha_\varphi^2}{\sin^2\theta'}\right\}^{\frac{1}{2}} \\ &+ \int_{r_0}^{r} dr'\left\{2\mu[w - U(r')] - \frac{\alpha_\theta^2}{r'^2}\right\}^{\frac{1}{2}} \end{aligned} \tag{101}$$

(α_φ, α_θ integration constants). Then, from the relations

$$p_\varphi = \frac{\partial \mathcal{S}}{\partial \varphi}, \quad p_\theta = \frac{\partial \mathcal{S}}{\partial \theta}, \quad p_r = \frac{\partial \mathcal{S}}{\partial r}, \tag{102}$$

one easily verifies that α_φ and α_θ are the z component and the absolute value of the angular momentum in the centre-of-mass frame and that w is clearly the energy in the same frame (internal energy). Thus $\mathcal{S}$ can be rewritten in terms of P_4, P_5, P_6 as

$$\begin{aligned} \mathcal{S}(r, \varphi, \theta, P_4, P_5, P_6) = P_4\varphi &+ \int_{\theta_0}^{\theta} d\theta'\left[P_5 - \frac{P_4^2}{\sin^2\theta'}\right]^{\frac{1}{2}} \\ &+ \int_{r_0}^{r} dr'\left\{-2\mu[P_6 + U(r')] - \frac{P_5}{r'^2}\right\}^{\frac{1}{2}}, \end{aligned} \tag{103}$$

so that there follows

$$\begin{aligned} Q_4 &= \frac{\partial \mathcal{S}}{\partial P_4} = \varphi - \int_{\theta_0}^{\theta} d\theta' \frac{P_4/\sin^2\theta'}{[P_5 - P_4^2/\sin^2\theta']^{\frac{1}{2}}}, \\ Q_5 &= \frac{\partial \mathcal{S}}{\partial P_5} = \frac{1}{2}\int_{\theta_0}^{\theta} d\theta'[P_5 - P_4^2/\sin^2\theta']^{-\frac{1}{2}} \\ &\quad - \frac{1}{2}\int_{r_0}^{r}\frac{dr'}{r'^2}\left\{-2\mu[P_6 + U(r')] - \frac{P_5}{r'^2}\right\}^{-\frac{1}{2}}, \\ Q_6 &= \frac{\partial \mathcal{S}}{\partial P_6} = -\mu\int_{r_0}^{r} dr'\left\{-2\mu[P_6 + U(r')] - \frac{P_5}{r'^2}\right\}^{-\frac{1}{2}}. \end{aligned} \tag{104}$$

Evaluating the first integral, one obtains the expected expression for Q_4, namely,

$$\begin{aligned} Q_4 &= \arctan\frac{\cos\varphi p_\theta - \cot\theta\sin\varphi p_\varphi}{-\sin\varphi p_\theta - \cot\theta\cos\varphi p_\varphi} \\ &= \arctan\frac{S_y}{S_x}. \end{aligned} \tag{105}$$

The second and third integrals can obviously be evaluated only for a definite choice of the interaction

[6] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon Press Inc., New York, 1960), Chap. VII, Sec. 48.

$U(r)$. For instance, in the case of free particles $U(r) \equiv 0$, one gets

$$Q_5 = \frac{1}{2S} \arctan \frac{p_\theta \tan\theta}{S} - \frac{1}{2S} \arctan \frac{rp_r}{S}$$
$$= \frac{1}{2S} \arctan \frac{p_\theta \tan\theta - rp_r}{S + \dfrac{rp_r p_\theta \tan\theta}{S}}, \tag{106}$$

$$Q_6 = \frac{rp_r}{2P_6} = -\frac{\mu r^3 p_r}{r^2 p_r^2 + S^2},$$

which are to be compared with the same expressions derived in Appendix C. In the case of the Coulomb potential $U(r) = -g/r$, one gets, instead,

$$Q_5 = \frac{1}{2S} \arctan \frac{p_\theta \tan\theta}{S} - \frac{1}{2S} \arctan \frac{rp_r}{S - \dfrac{g\mu r}{S}},$$

$$Q_6 = \frac{rp_r}{2P_6} + \frac{g\mu}{2P_6[2\mu P_6]^{\frac{1}{2}}} \arctan \frac{g\mu - 2\mu r P_6}{rp_r[2\mu P_6]^{\frac{1}{2}}}, \tag{107}$$

where

$$-P_6 \equiv -\mathfrak{I}_2 = W = \frac{p_r^2}{2\mu} + \frac{\mathbf{S}^2}{2\mu r^2} - \frac{g}{r}. \tag{108}$$

Let us remark that the first term in the expression of the variable Q_5, for any interaction, coincides with the corresponding expression of the same variable in the case of the rigid rod. It is also clear that the above treatment contains both the bound motion case and the unbound one through the sign of $P_6 \equiv \mathfrak{I}_2$.

In a forthcoming paper we shall discuss in detail the canonical realizations of the inhomogeneous Lorentz group.

APPENDIX A: SOLUTIONS OF THE SYSTEM (34)

The system can be written more concisely as

$$\epsilon_{ijl} \frac{\partial\Phi}{\partial M_j} K_l - m \frac{\partial\Phi}{\partial T_i} + \frac{\partial\Phi}{\partial E} T_i = 0,$$
$$\epsilon_{ijl} \frac{\partial\Phi}{\partial M_j} T_l + m \frac{\partial\Phi}{\partial K_i} = 0, \quad i = x, y, z. \tag{A1}$$

This is a linear complete homogeneous system of six equations in ten variables, which must have four independent solutions.

First of all, let us look for a particular solution $\Phi = (\mathbf{T}, \mathbf{K}, E)$, independent of $\mathbf{M}$. It follows that

$$\frac{\partial\Phi}{\partial E} T_i - m \frac{\partial\Phi}{\partial T_i} = 0, \tag{A2}$$

$$\frac{\partial\Phi}{\partial K_i} = 0 \quad (i = x, y, z). \tag{A3}$$

Using the method of the characteristics already used in II (Appendix D), we can write

$$\frac{1}{T_x}\frac{\partial E}{\partial T_x} = \frac{1}{T_y}\frac{\partial E}{\partial T_y} = \frac{1}{T_z}\frac{\partial E}{\partial T_z} = -\frac{1}{m}, \tag{A4}$$

from which we get

$$E = -\frac{1}{2m}(T_x^2 + T_y^2 + T_z^2) + \Phi_0. \tag{A5}$$

Thus, the solution we were looking for can be written

$$\Phi_0 = \frac{\mathbf{T}^2}{2m} + E, \tag{A6}$$

which coincides with the expression (36). Then, let us search for three other independent solutions, which are functions of only $\mathbf{M}$, $\mathbf{T}$, and $\mathbf{K}$. Direct inspection of (A1) suggests a need for expressions linear in $\mathbf{M}$ of the form

$$\Phi = \mathbf{a} \cdot \mathbf{M} + \Psi(\mathbf{K}, \mathbf{T}), \tag{A7}$$

$\mathbf{a}$ being a constant vector. Such an expression, when replaced in (A1), leads to the total differential system

$$\epsilon_{ijl} a_j K_l - m \frac{\partial\Psi}{\partial T_i} = 0,$$
$$\epsilon_{ijl} a_j T_l + m \frac{\partial\Psi}{\partial K_i} = 0, \tag{A8}$$

which gives

$$\Psi = \frac{1}{m} \epsilon_{ijl} a_j T_i K_l + \text{const.} \tag{A9}$$

Consequently, we obtain

$$\Phi = \mathbf{a} \cdot \mathbf{M} + \frac{1}{m} \mathbf{a} \cdot \mathbf{K} \times \mathbf{T}, \tag{A10}$$

and finally, choosing $\mathbf{a}$ as the unit vector along the three coordinate axes, we get the expressions (35).

APPENDIX B: CONSTRUCTION OF SCHEME A FOR THE *REGULAR* REALIZATIONS WITH $m = 0$

As stated in Sec. 3, we have also in this case $h = 4$, $k = 2$. Let us start with choosing $\mathfrak{P}_1 = M_z$. Then, the results of II (Sec. 2) enable us to put $\mathfrak{Q}_1 = \arctan M_y/M_x$. The system

$$\{\mathfrak{P}_1, \Phi\} = 0,$$
$$\{\mathfrak{Q}_1, \Phi\} = 0 \tag{B1}$$

is complete and admits eight independent solutions. It is evident that any scalar which can be constructed with the generators is a solution of the system. It can be checked that a possible choice of the independent scalars is

$$\mathbf{M}^2,\ \mathbf{T}^2,\ \mathbf{M}\cdot\mathbf{T},\ \mathbf{K}\cdot\mathbf{T},\ E,\ \Gamma \equiv |\mathbf{K}\times\mathbf{T}|^2,$$
$$\Lambda \equiv \mathbf{K}\times\mathbf{T}\cdot\mathbf{M}. \tag{B2}$$

A further independent solution remains to be determined, which, obviously, cannot be a scalar. Since M_z, T_z, and the scalars (B2) are nine independent variables which have a zero Poisson bracket with $\mathfrak{P}_1$, in order to find the remaining solution it is sufficient to look for a function Φ of such variables having a zero Poisson bracket with $\mathfrak{Q}_1$. We must have

$$\{\mathfrak{Q}_1, \Phi\} \equiv \frac{\mathbf{M}\cdot\mathbf{T} - M_z T_z}{M^2 - M_z^2}\frac{\partial\Phi}{\partial T_z} + \frac{\partial\Phi}{\partial M_z} = 0, \quad \text{(B3)}$$

$$M = |\mathbf{M}|.$$

Thus, Φ can be assumed to be a function of only M_z, T_z, M^2, and $\mathbf{M}\cdot\mathbf{T}$. Solving (B3) with the method of the characteristics, we find

$$\Phi = \frac{T_z - \dfrac{\mathbf{M}\cdot\mathbf{T}}{M^2}M_z}{[M^2 - M_z^2]^{\frac{1}{2}}} = \frac{[(\mathbf{M}\times\mathbf{T})\times\mathbf{M}]_z}{M^2[M^2 - M_z^2]^{\frac{1}{2}}}. \quad \text{(B4)}$$

Now, we can choose $\mathfrak{P}_2 = \mathbf{M}^2$ and look for a function $\mathfrak{Q}_2$ of $\mathbf{M}^2$, Φ, and the remaining variables, such that

$$\{\mathfrak{Q}_2, \mathbf{M}^2\} = 1. \quad \text{(B5)}$$

It holds that

$$\{\Phi, \mathbf{M}^2\} = 2\frac{M_x T_y - M_y T_x}{[M^2 - M_z^2]^{\frac{1}{2}}} = 2\frac{[\mathbf{M}\times\mathbf{T}]_z}{[M^2 - M_z^2]^{\frac{1}{2}}}, \quad \text{(B6)}$$

while, obviously, all the scalars (B2) have zero Poisson brackets with $\mathbf{M}^2$. As a consequence of the Jacobi identity, the expression (B6) must be a solution of the system (B1) [see I (Sec. 2)], and so it could be reexpressed in terms of Φ and of the variables (B2). It is more convenient to denote the quantity (B6) by $2\Phi'$ and to notice that it holds true for

$$\{\Phi, \mathbf{M}^2\} = 2\Phi',$$
$$\{\Phi', \mathbf{M}^2\} = -2M^2\Phi. \quad \text{(B7)}$$

Then, $\mathfrak{Q}_2$ can be assumed to be a function of Φ, Φ', M^2, and (B5) becomes

$$2\Phi'\frac{\partial\mathfrak{Q}_2}{\partial\Phi} - 2M^2\Phi\frac{\partial\mathfrak{Q}_2}{\partial\Phi'} = 1. \quad \text{(B8)}$$

A solution of the associate homogeneous equation is readily found to be

$$\Omega = \Phi'^2 + M^2\Phi^2. \quad \text{(B9)}$$

Then, using as independent variables Φ' and Ω instead of Φ' and Φ, we get

$$-2M(\Omega - \Phi'^2)^{\frac{1}{2}}\frac{\partial\mathfrak{Q}_2}{\partial\Phi'} = 1, \quad \text{(B10)}$$

from which it easily follows that

$$\mathfrak{Q}_2 = \frac{1}{2M}\arctan\left(-\frac{\Phi'}{M\Phi}\right)$$
$$= \frac{1}{2M}\arctan\frac{M[\mathbf{T}\times\mathbf{M}]_z}{M^2T_z - \mathbf{M}\cdot\mathbf{T}M_z}. \quad \text{(B11)}$$

At this point we have to look for six independent functions of the variables (B2) and of Φ which have zero Poisson brackets with $\mathfrak{P}_2$ and $\mathfrak{Q}_2$.

From the structure of $\mathfrak{Q}_2$, one can readily check that, among the variables (B2),

$$\mathbf{T}^2, \mathbf{M}\cdot\mathbf{T}, \mathbf{K}\cdot\mathbf{T}, E, \Gamma \quad \text{(B12)}$$

have the above property, while for Λ the following is true:

$$\{\mathfrak{Q}_2, \Lambda\} = -\frac{\Lambda\Gamma^2}{2[M^2T^2 - (\mathbf{M}\cdot\mathbf{T})^2]}, \quad T = |\mathbf{T}|. \quad \text{(B13)}$$

A sixth expression having zero Poisson brackets with $\mathfrak{P}_2$ and $\mathfrak{Q}_2$ can be obtained by

$$\{\mathfrak{Q}_2, \Psi(M^2, T^2, \mathbf{M}\cdot\mathbf{T}, \mathbf{K}\cdot\mathbf{T}, E, \Gamma, \Lambda)\}$$
$$\equiv \frac{\partial\Psi}{\partial M^2} - \frac{\Lambda\Gamma^2}{2[M^2T^2 - (\mathbf{M}\cdot\mathbf{T})^2]}\frac{\partial\Psi}{\partial\Lambda} = 0. \quad \text{(B14)}$$

A solution of this equation is

$$\Psi = \Lambda[M^2T^2 - (\mathbf{M}\cdot\mathbf{T})^2]^{\frac{1}{2}}. \quad \text{(B15)}$$

Then, setting $\mathfrak{P}_3 = \mathbf{M}\cdot\mathbf{T}$, with the usual procedure we find

$$\mathfrak{Q}_3 = \frac{1}{T}\arctan\frac{T(\mathbf{K}\times\mathbf{T}\cdot\mathbf{M})}{(\mathbf{K}\times\mathbf{T})\cdot(\mathbf{M}\times\mathbf{T})}. \quad \text{(B16)}$$

We are left with $\mathfrak{P}_4$ and $\mathfrak{Q}_4$. It is readily seen that $\mathbf{T}^2$, $\mathbf{K}\cdot\mathbf{T}$, and Γ have zero Poisson bracket with $\mathfrak{Q}_3$. The same is true for E, since it is obvious that it has zero Poisson bracket with $\mathbf{K}\times\mathbf{T}$. Thus, if one sets $\mathfrak{P}_4 = E$, an obvious choice for $\mathfrak{Q}_4$ is

$$\mathfrak{Q}_4 = \frac{\mathbf{K}\cdot\mathbf{T}}{T^2}. \quad \text{(B17)}$$

The remaining quantities $\mathbf{T}^2$ and Γ can be directly assumed as the two independent canonical *invariants.*

APPENDIX C: DETERMINATION OF THE VARIABLES Q_5 AND Q_6 FOR THE SYSTEM OF TWO FREE PARTICLES ACCORDING TO THE CONSTRUCTIVE PROCEDURE OF THEOREM 2 OF I

The Variable Q_5: We have to search for a function $Q_5 = Q_5(r, p_r, \varphi, \theta)$ such that

$$\{Q_4, Q_5\} = \{P_4, Q_5\} = 0,$$
$$\{Q_5, \mathbf{S}^2\} = 1, \quad \{Q_5, P_6\} = 0. \quad \text{(C1)}$$

The result obtained for the rigid body suggests looking for an expression of the form

$$Q_5 = \frac{1}{2S}\arctan\frac{p_\theta \tan\theta}{S} + \Phi(r, p_r, S^2). \quad \text{(C2)}$$

The first three relations (C1) are then automatically satisfied. The last one gives

$$\{\Phi, \mathfrak{I}_2\} - \frac{1}{2\mu r^2} \equiv -\frac{1}{2\mu r^2} - \frac{p_r}{\mu}\frac{\partial \Phi}{\partial r} - \frac{S^2}{\mu r^3}\frac{\partial \Phi}{\partial p_r} = 0. \tag{C3}$$

Putting $\xi \equiv rp_r$, $\eta \equiv r$, this becomes

$$(\xi^2 + S^2)\frac{\partial \Phi}{\partial \xi} + \xi\eta\frac{\partial \Phi}{\partial \eta} + \frac{1}{2} = 0. \tag{C4}$$

A particular solution of Eq. (C4) is easily found by assuming Φ to be independent of η, and the following is obtained:

$$(\xi^2 + S^2)\frac{\partial \Phi}{\partial \xi} = -\tfrac{1}{2}, \tag{C5}$$

and so

$$\Phi = -\frac{1}{2S}\arctan\frac{rp_r}{S} + \text{const.} \tag{C6}$$

In conclusion we have

$$Q_5 = \frac{1}{2S}\arctan\frac{p_\theta \tan\theta}{S} - \frac{1}{2S}\arctan\frac{rp_r}{S} = \frac{1}{2S}\arctan\frac{p_\theta\tan\theta - rp_r}{S + \dfrac{rp_rp_\theta\tan\theta}{S}}. \tag{C7}$$

The Variable Q_6: The variable Q_6 is easily determined by observing that two independent functions which commute with both P_5 and Q_5 are given by $\xi = rp_r$ and $\mathfrak{I}_2$ itself. Then, for a function $\Psi = \Psi(\xi, \mathfrak{I}_2)$, we have

$$\{\Psi, \mathfrak{I}_2\} = 1, \tag{C8}$$

i.e.,

$$\frac{\partial \Psi}{\partial \xi}\left(-\frac{p_r^2}{\mu} - \frac{S^2}{\mu r^2}\right) = 2\mathfrak{I}_2\frac{\partial \Psi}{\partial \xi} = 1. \tag{C9}$$

Thus, we conclude that

$$Q_6 = \frac{1}{2\mathfrak{I}_2} rp_r. \tag{C10}$$

TABLE I. The variables of the *typical form* (Scheme B) for the physical ($m > 0$) canonical realizations discussed in the present paper.

Free mass point

($\mathfrak{I}_1 = 0$, $\mathfrak{I}_2 = \text{const}$).

$$\text{I}\quad \begin{cases} P_1 = p_x & Q_1 = q_x \\ P_2 = p_y & Q_2 = q_y \\ P_3 = p_z & Q_3 = q_z \end{cases}$$

Free particle with spin

($\mathfrak{I}_1 = s^2$, $\mathfrak{I}_2 = \text{const}$).

$$\text{I}\quad \begin{cases} P_1 = p_x & Q_1 = q_x \\ P_2 = p_y & Q_2 = q_y \\ P_3 = p_z & Q_3 = q_z \\ P_4 = p_\chi & Q_4 = \chi \end{cases}$$

Rigid rod ($\mathfrak{I}_2' \equiv \mathfrak{I}_2 + \frac{1}{2I}\mathfrak{I}_1 = \text{const}$).

$$\text{I}\quad \begin{cases} P_1 = p_x & Q_1 = q_x \\ P_2 = p_y & Q_2 = q_y \\ P_3 = p_z & Q_3 = q_z \\ P_4 = p_\varphi & Q_4 = \arctan\dfrac{\cos\varphi p_\theta - \cot\theta\sin\varphi p_\varphi}{-\sin\varphi p_\theta - \cot\theta\cos\varphi p_\varphi} \end{cases}$$

$$\text{II}\quad P_5 = p_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2 \qquad Q_5 = \frac{1}{2(P_5)^{\frac{1}{2}}}\arctan\frac{p_\theta\tan\theta}{(P_5)^{\frac{1}{2}}}$$

Spherical top ($\mathfrak{I}_2' \equiv \mathfrak{I}_2 + \frac{1}{2I}\mathfrak{I}_1 = \text{const}$)[a].

$$\text{I}\quad \begin{cases} P_1 = p_x & Q_1 = q_x \\ P_2 = p_y & Q_2 = q_y \\ P_3 = p_z & Q_3 = q_z \\ P_4 = p_\varphi & Q_4 = \arctan\dfrac{\sin\varphi p_\theta - \dfrac{\cos\varphi}{\sin\theta}p_\psi + \cot\theta\cos\varphi p_\varphi}{\cos\varphi p_\theta + \dfrac{\sin\varphi}{\sin\theta}p_\psi - \cot\theta\sin\varphi p_\varphi} \end{cases}$$

[a] Note that, looking at Scheme B for the spherical top, the part corresponding to the internal motion can be considered as providing a particular class of realizations (viz., the "symmetrical" ones) of the group O(3) ⊗ O(3), which is an "invariance" dynamical symmetry group for the intrinsic motion of the system. This is no longer true for the symmetrical and asymmetrical top, in which the dynamical symmetry is broken.

TABLE I (*contd.*)

II	$P_5 = p_\theta + \frac{1}{\sin^2\theta}(p_\varphi^2 + p_\psi^2) - \frac{2\cot\theta}{\sin\theta} p_\varphi p_\psi$	$Q_5 = \frac{1}{2(P_5)^{\frac{1}{2}}} \arctan \frac{p_\theta \tan\theta}{P_5^{\frac{1}{2}} - \frac{p_\varphi p_\psi}{(P_5)^{\frac{1}{2}} \cos\theta}}$
IV	$P_6 = p_\psi$	$Q_6 = \arctan \frac{\sin\psi p_\theta - \frac{\cos\psi}{\sin\theta} p_\varphi + \cot\theta \cos\psi p_\psi}{\cos\psi p_\theta + \frac{\sin\psi}{\sin\theta} p_\varphi - \cot\theta \sin\psi p_\psi}$

Symmetrical top (1st choice, cf. Eqs. (72), (82), (84)).

I	$P_1 = p_x$	$Q_1 = q_x$
	$P_2 = p_y$	$Q_2 = q_y$
	$P_3 = p_z$	$Q_3 = q_z$
	$P_4 = p_\varphi$	$Q_4 = \arctan \frac{\sin\varphi p_\theta - \frac{\cos\varphi}{\sin\theta} p_\psi + \cot\theta \cos\varphi p_\varphi}{\cos\varphi p_\theta + \frac{\sin\varphi}{\sin\theta} p_\psi - \cot\theta \sin\varphi p_\varphi}$
II	$P_5 = p^2 + \frac{1}{\sin^2\theta}(p_\varphi^2 + p_\psi^2) - \frac{2\cot\theta}{\sin\theta} p_\varphi p_\psi$	$Q_5 = \frac{1}{2(P_5)^{\frac{1}{2}}} \arctan \frac{p_\theta \tan\theta}{(P_5)^{\frac{1}{2}} - \frac{p_\varphi p_\psi}{(P_5)^{\frac{1}{2}} \cos\theta}}$
	$P_6 = \frac{I_3 - I}{2II_3} p_\psi^2$	$Q_6 = \frac{II_3}{I_3 - I} \frac{1}{p_\psi} \arctan \frac{\sin\psi p_\theta - \frac{\cos\psi}{\sin\theta} p_\varphi + \cot\theta \cos\psi p_\psi}{\cos\psi p_\theta + \frac{\sin\psi}{\sin\theta} p_\varphi - \cot\theta \sin\psi p_\psi}$

System of two free mass points

I	$P_1 = P_x$	$Q_1 = \frac{1}{m} \sum_i m_i q_{i_x}$
	$P_2 = P_y$	$Q_2 = \frac{1}{m} \sum_i m_i q_{i_y}$
	$P_3 = P_z$	$Q_3 = \frac{1}{m} \sum_i m_i q_{i_z}$
	$P_4 = p_\varphi$	$Q_4 = \arctan \frac{\cos\varphi p_\theta - \cot\theta \sin\varphi p_\varphi}{-\sin\varphi p_\theta - \cot\theta \cos\varphi p_\varphi}$
II	$P_5 = p_\theta^2 + \frac{1}{\sin^2\theta} p_\varphi^2$	$Q_5 = \frac{1}{2(P_5)^{\frac{1}{2}}} \arctan \frac{p_\theta \tan\theta}{(P_5)^{\frac{1}{2}}} - \frac{1}{2(P_5)^{\frac{1}{2}}} \arctan \frac{r p_r}{(P_5)^{\frac{1}{2}}}$
	$P_6 = -\frac{p_r^2}{2\mu} - \frac{S^2}{2\mu r^2}$	$Q_6 = \frac{1}{2P_6} r p_r$

System of two mass points interacting through a Coulomb potential: $U(r) = -g/r$.

I	$P_1 = P_x$	$Q_1 = \frac{1}{m} \sum_i m_i q_{i_x}$
	$P_2 = P_y$	$Q_2 = \frac{1}{m} \sum_i m_i q_{i_y}$
	$P_3 = P_z$	$Q_3 = \frac{1}{m} \sum_i m_i q_{i_z}$
	$P_4 = p_\varphi$	$Q_4 = \arctan \frac{\cos\varphi p_\theta - \cot\theta \sin\varphi p_\varphi}{-\sin\varphi p_\theta - \cot\theta \cos\varphi p_\varphi}$
II	$P_5 = p_\theta^2 + \frac{1}{\sin^2\theta} p_\varphi^2$	$Q_5 = \frac{1}{2(P_5)^{\frac{1}{2}}} \arctan \frac{p_\theta \tan\theta}{(P_5)^{\frac{1}{2}}} - \frac{1}{2(P_5)^{\frac{1}{2}}} \arctan \frac{r p_r}{(P_5)^{\frac{1}{2}} - \frac{g\mu r}{(P_5)^{\frac{1}{2}}}}$
	$P_6 = \frac{g}{r} - \frac{p_r^2}{2\mu} - \frac{S^2}{2\mu r^2}$	$Q_6 = \frac{r p_r}{2P_6} + \frac{g\mu}{(8\mu P_6^3)^{\frac{1}{2}}} \arctan \frac{g\mu - 2\mu r P_6}{r p_r (2\mu P_6)^{\frac{1}{2}}}$

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Special Functions and the Complex Euclidean Group in 3-Space. I

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It is shown that the general addition theorems of Gegenbauer, relating Bessel functions and Gegenbauer polynomials, are special cases of identities for special functions obtained from a study of certain local irreducible representations of the complex Euclidean group in 3-space. Among the physically interesting results generalized by this analysis are the expansion for a plane wave as a sum of spherical waves and the addition theorem for spherical waves. This paper is one of a series attempting to derive the special functions of mathematical physics and their basic properties from the representation theory of Lie symmetry groups.

INTRODUCTION

The cylindrical (Bessel) functions obey two distinct types of addition theorems: those of Graf and Gegenbauer.[1] Graf's addition theorems are closely related to the representation theory of the Euclidean group in the plane and are obtained from a study of the solutions of the wave equation in 2-space.[2–4] On the other hand, the addition theorems of Gegenbauer are usually considered as by-products of the representation theory of the Euclidean group in n-space and are ordinarily derived from a study of the wave equation in n-space. It will be shown, however, that the Gegenbauer theorems can be derived (and even extended) from a study of certain representations of the Euclidean group in 3-space alone.

The results presented here are part of a continuing program by the author to uncover the relationship between Lie symmetry groups and the special functions of mathematical physics.[5,6] In this program, symmetry groups are considered as fundamental objects, while special functions and their properties are derived in a systematic fashion from the representation theory of the symmetry groups. The special functions associated with a given group arise in two ways: as matrix elements corresponding to a representation of the group, and as basis vectors in a model of such a representation. To the extent that matrix elements and models can be derived systematically for a given group, a large part of special function theory can be derived systematically from the theory of Lie groups.

In this paper, we examine a restricted class of irreducible representations of the complex Euclidean group in 3-space and obtain identities relating Bessel functions and Gegenbauer polynomials. In future papers, we shall examine other representations of this group and derive identities relating Whittaker functions and Jacobi polynomials.

1. REPRESENTATIONS OF THE EUCLIDEAN GROUP

We denote by $\mathcal{T}_6$ the 6-dimensional complex Lie algebra with generators p^+, p^-, p^3, j^+, j^-, and j^3 commutation relations as follows:

$$\begin{aligned}
&[j^3, j^\pm] = \pm j^\pm, \ [j^+, j^-] = 2j^3,\\
&[j^3, p^\pm] = [p^3, j^\pm] = \pm p^\pm,\\
&[j^+, p^+] = [j^-, p^-] = [j^3, p^3] = \mathcal{O},\\
&[j^+, p^-] = [p^+, j^-] = 2p^3,\\
&[p^3, p^\pm] = [p^+, p^-] = \mathcal{O}.
\end{aligned} \tag{1.1}$$

The elements j^+, j^-, j^3 generate a subalgebra of $\mathcal{T}_6$ isomorphic to $sl(2)$, the Lie algebra of 2×2 traceless matrices.[6] The elements p^+, p^-, p^3 generate a 3-dimensional Abelian ideal in $\mathcal{T}_6$.

Denote by T_6 the complex 6-parameter Lie group consisting of all elements $\{\mathbf{w}, g\}$,

$$\mathbf{w} = (\alpha, \beta, \gamma) \in \mathbb{C}^3, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2),$$
$$ad - bc = 1, \tag{1.2}$$

with group multiplication

$$\{\mathbf{w}, g\}\{\mathbf{w}', g\} = \{\mathbf{w} + g\mathbf{w}', gg'\}, \tag{1.3}$$

where "+" denotes vector addition in $\mathbb{C}^3$ and

$$\begin{aligned} g\mathbf{w} = (a^2\alpha - b^2\beta + ab\gamma,\ &-c^2\alpha + d^2\beta - cd\gamma,\\ &2ac\alpha - 2bd\beta + (bc + ad)\gamma). \end{aligned} \tag{1.4}$$

Here $\mathbf{w}$ is a complex 3-vector and g is a complex 2×2 unimodular matrix. The identity element of

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[1] W. Magnus, F. Oberhettinger, and R. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966), 3rd ed.

[2] N. Y. Vilenkin, Usp. Mat. Nauk. N.S., **11**, No. 3, 69 (1956).

[3] N. Y. Vilenkin, *Special Functions and Theory of Group Representations* (Izd. Nauka., Moscow, 1965).

[4] E. P. Wigner, *The Application of Group Theory to the Special Functions of Mathematical Physics*, Princeton Lecture Notes (1955) (unpublished).

[5] W. Miller, *On Lie Algebras and Some Special Functions of Mathematical Physics*, American Mathematical Society Memoir, No. 50 (Providence, 1964).

[6] W. Miller, *Lie Theory and Special Functions* (Academic Press Inc., New York, 1968), Chaps. 5, 6.

T_6 is $\{\mathbf{0}, \mathbf{e}\}$, where $\mathbf{0} = (0, 0, 0)$ and $\mathbf{e}$ is the identity element of $SL(2)$, and the inverse of an element $\{\mathbf{w}, g\}$ is given by

$$\{\mathbf{w}, g\}^{-1} = \{-g^{-1}\mathbf{w}, g^{-1}\}.$$

The set of all elements $\{\mathbf{0}, g\}$, $g \in SL(2)$, forms a subgroup of T_6 which can be identified with $SL(2)$. Similarly, the set of all elements $\{\mathbf{w}, \mathbf{e}\}$, $\mathbf{w} \in \mathcal{C}^3$, forms a subgroup of T_6 which can be identified with $\mathcal{C}^3$.

It is straightforward to show that $\mathcal{T}_6$ is the Lie algebra of T_6. Indeed, the generators of $\mathcal{T}_6$ can be chosen so that

$$\{\mathbf{w}, g\} = \exp(\alpha p^+ + \beta p^- + \gamma p^3) \exp[(-b/d)j^+] \times \exp(-cdj^-) \exp(-2 \ln dj^3), \quad (1.5)$$

where $\{\mathbf{w}, g\}$ is defined by Eq. (1.2) and g is in a sufficiently small neighborhood of $\mathbf{e}$ [in the topology of $SL(2)$].[6] Here "exp" is the exponential map of a neighborhood of $\mathcal{O}$ in $\mathcal{T}_6$ onto a neighborhood of $\{\mathbf{o}, \mathbf{e}\}$ in T_6.[7]

The complex group T_6 is closely related to the real Euclidean group in 3-space[6]: the set of all pairs $(\mathbf{r}, R)$, $\mathbf{r}$ a real 3-vector, R a proper 3×3 orthogonal matrix, with group multiplication

$$(\mathbf{r}, R)(\mathbf{r}', R') = (\mathbf{r} + R\mathbf{r}', RR').$$

To see this we note that E_6, the real, simply connected covering group of the Euclidean group, can be defined as the set of all pairs $(\mathbf{r}, A)$, where $\mathbf{r} = (r_1, r_2, r_3)$ is a real column vector and A is an element of $SU(2)$ (the group of 2×2 unitary unimodular matrices). The group multiplication law is

$$(\mathbf{r}, A)(\mathbf{r}', A') = [\mathbf{r} + R(A)\mathbf{r}', AA'],$$

where $R(A)$ is a real 3×3 orthogonal matrix given explicitly by

$$R(A) = \begin{pmatrix} \frac{1}{2}(a^2-b^2+\bar{a}^2-\bar{b}^2), & \frac{i}{2}(\bar{a}^2+\bar{b}^2-a^2-b^2), & \bar{a}\bar{b}+ab \\ \frac{i}{2}(a^2-b^2-\bar{a}^2+\bar{b}^2), & \frac{1}{2}(\bar{a}^2+\bar{b}^2+a^2+b^2), & i(-\bar{a}\bar{b}+ab) \\ -(\bar{a}b+a\bar{b}), & i(-\bar{a}b+a\bar{b}), & a\bar{a}-b\bar{b} \end{pmatrix},$$

when

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \in SU(2), \quad a\bar{a} + b\bar{b} = 1.$$

Now, E_6 can be considered as a real subgroup of T_6. Indeed, it is easy to show that the collection of all elements $\{\mathbf{w}, A\}$, where $\mathbf{w} = [\frac{1}{2}(-r_2 - ir_1), \frac{1}{2}(r_2 - ir_1), -ir_3]$ and $A \in SU(2)$ forms a subgroup of T_6 isomorphic to E_6. The isomorphism is given by $(\mathbf{r}, A) \leftrightarrow \{\mathbf{w}, A\}$, $\mathbf{r} = (r_1, r_2, r_3)$.

The real 6-dimensional Lie algebra $\mathcal{E}_6$ corresponding to E_6 is generated by elements j_k, p_k, $k = 1, 2, 3$, with commutation relations

$$[j_j, j_k] = \epsilon_{jkl} j_l, \quad [j_j, p_k] = \epsilon_{jkl} p_l,$$
$$[p_j, p_k] = \mathcal{O}, \qquad j, k, l = 1, 2, 3, \quad (1.6)$$

where ϵ_{jkl} is the completely antisymmetric tensor such that $\epsilon_{123} = +1$. We choose these generators so that they are related to the finite group elements by

$$(\mathbf{r}, A) = \exp(r_1 p_1 + r_2 p_2 + r_3 p_3) \exp \varphi_1 j_3 \times \exp \theta j_1 \exp \varphi_2 j_3,$$

where $\mathbf{r} = (r_1, r_2, r_3)$ and φ_1, θ, φ_2 are the Euler coordinates for A. The formal elements $p^\pm$, p^3, $j^\pm$, j^3, defined in terms of the generators (1.6) by

$$p^\pm = \mp p_2 + ip_1, \quad p^3 = ip_3,$$
$$j^\pm = \pm j_2 + ij_1, \quad j^3 = ij_3,$$

can easily be shown to satisfy the commutation relations (1.1) for the complex Lie algebra $\mathcal{T}_6$. Thus, we have explicitly determined $\mathcal{E}_6$ as a real form of $\mathcal{T}_6$ and $\mathcal{T}_6$ as a complexification of $\mathcal{E}_6$. In this sense we can say that T_6 is a complexification of the Euclidean group in 3-space.

Consider a complex vector space V (possibly infinite-dimensional) and a representation ρ of $\mathcal{T}_6$ by linear operators on V.[5,6] Set

$$\rho(p^\pm) = P^\pm, \quad \rho(p^3) = P^3,$$
$$\rho(j^\pm) = J^\pm, \quad \rho(j^3) = J^3.$$

Then the linear operators $P^\pm$, P^3, $J^\pm$, J^3 satisfy commutation relations on V analogous to Eq. (1.1), where now $[A, B] = AB - BA$ for operators A and B on V. We define two operators on V which are of special importance for the representation theory of $\mathcal{T}_6$. They are

$$\mathbf{P} \cdot \mathbf{P} = -P^+P^- - P^3P^3,$$
$$\mathbf{P} \cdot \mathbf{J} = \tfrac{1}{2}(P^+J^- + P^-J^+) - P^3J^3. \quad (1.7)$$

It is easy to show that

$$[\mathbf{P} \cdot \mathbf{P}, \rho(\alpha)] = [\mathbf{P} \cdot \mathbf{J}, \rho(\alpha)] = 0$$

for all $\alpha \in \mathcal{T}_6$. Thus, if ρ is an irreducible representation of $\mathcal{T}_6$, we would expect $\mathbf{P} \cdot \mathbf{P}$ and $\mathbf{P} \cdot \mathbf{J}$ to be multiples of the identity operator on V.

The irreducible representations of $\mathcal{T}_6$ which are of interest in special function theory have been classified.[5,6] Among these representations we single out the following two classes related to Gegenbauer polynomials and Bessel functions:

(1) $\rho_0(\omega)$.

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u-1, \cdots, -u+1, -u$, and $u = 0, 1, 2, \cdots$.

(2) $\rho_\mu(\omega)$, ($0 \leq \operatorname{Re} \mu < 1$ and 2μ not an integer).

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u-1, u-2, \cdots$, and $u = \mu + n$, where $n = 0, \pm 1, \pm 2, \cdots$.

These representations are defined for any nonzero complex number ω. Furthermore, corresponding to each representation, the action of the infinitesimal

[7] S. Helgason, *Differential Geometry and Symmetric Spaces* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1962), Chap. 2.

operators on the basis vectors $f_m^{(u)}$ is given by

$$J^3 f_m^{(u)} = m f_m^{(u)}, \quad J^+ f_m^{(u)} = (m-u) f_{m+1}^{(u)},$$
$$J^- f_m^{(u)} = -(m+u) f_{m-1}^{(u)}, \tag{1.8}$$

$$P^3 f_m^{(u)} = \frac{\omega}{2u+1} f_m^{(u+1)} + \frac{\omega(u+m)(u-m)}{2u+1} f_m^{(u-1)}, \tag{1.9}$$

$$P^+ f_m^{(u)} = \frac{\omega}{2u+1} f_{m+1}^{(u+1)} - \frac{\omega(u-m)(u-m-1)}{2u+1} f_{m+1}^{(u-1)}, \tag{1.10}$$

$$P^- f_m^{(u)} = \frac{-\omega}{2u+1} f_{m-1}^{(u+1)} + \frac{\omega(u+m)(u+m-1)}{2u+1} f_{m-1}^{(u-1)}, \tag{1.11}$$

$$\mathbf{P}\cdot\mathbf{P} f_m^{(u)} = -\omega^2 f_m^{(u)}, \quad \mathbf{P}\cdot\mathbf{J} f_m^{(u)} = 0. \tag{1.12}$$

[If a vector $f_m^{(u)}$ on the right-hand side of one of the expressions (1.8)–(1.12) does not belong to the representation space, we set this vector equal to zero.]

It is easy to verify directly that the infinitesimal operators given by these expressions (1.8)–(1.11) do satisfy the commutation relations (1.1) and define an irreducible representation of $\mathcal{C}_6$. Furthermore, the vectors $\{f_m^{(u)}\}$, corresponding to some fixed value of u, form a basis for an irreducible representation of the subalgebra $sl(2)$ of $\mathcal{C}_6$. Each such induced representation of $sl(2)$ associated with $\rho_0(\omega)$ has dimension $2u+1$ and is denoted by $D(2u)$. Each such induced representation of $sl(2)$ associated with $\rho_\mu(\omega)$ is infinite-dimensional and is denoted by $\downarrow u$. The representations $D(2u)$ and $\downarrow u$ have been studied in detail elsewhere.[5,6]

Our aim in this paper is to examine the relationship between the representations $\rho_0(\omega)$, $\rho_\mu(\omega)$ and special function theory. In particular, we shall be interested in the following two aspects of this relationship:

(1) We can look for models of the abstract representations $\rho_0(\omega)$, $\rho_\mu(\omega)$ such that the infinitesimal operators $\rho(\alpha)$, $\alpha \in \mathcal{C}_6$, are linear differential operators acting on a space V of analytic functions in n complex variables. In this case the basis vectors $f_m^{(u)}$ will be analytic functions and expressions (1.8)–(1.11) will yield differential recursion relations obeyed by these "special" functions. For $n = 1, 2$, all of the possible models have been constructed.[6] In particular, for $n = 1$ it is known that no models exist. For $n = 2$, there is Model A:

$$J^3 = t\frac{\partial}{\partial t}, \quad J^+ = -t\frac{\partial}{\partial z},$$
$$J^- = t^{-1}\left[(1-z^2)\frac{\partial}{\partial z} - 2zt\frac{\partial}{\partial t}\right], \tag{1.13}$$
$$P^+ = \omega t, \quad P^- = \omega(1-z^2)t^{-1}, \quad P^3 = \omega z.$$

Corresponding to this model, the basis vectors $f_m^{(u)}$ are uniquely defined by relations (1.8)–(1.12) up to an arbitrary multiplicative constant and may be given by

$$f_m^{(u)}(z,t) = \Gamma(u-m+1)\Gamma(m+\tfrac{1}{2})C_{u-m}^{m+\frac{1}{2}}(z)(2t)^m. \tag{1.14}$$

Here $\Gamma(x)$ is the gamma function and $C_n^\lambda(z)$ is a Gegenbauer polynomial defined by the generating function

$$(1-2\alpha z+\alpha^2)^{-\lambda} = \sum_{n=0} C_n^\lambda(z)\alpha^n.$$

If the representation under consideration is $\rho_0(\omega)$, then m takes the integer values $u, u-1, \cdots, -u$ and u runs over the nonnegative integers in Eq. (1.10). However, if the representation is $\rho_\mu(\omega)$, then $m = u, u-1, u-2, \cdots$, and u takes all values such that $u-\mu$ is an integer. Substitution of Eq. (1.13) and (1.14) into expressions (1.8)–(1.11) leads to some well-known recursion relations for the Gegenbauer polynomials:

$$\frac{d}{dz}C_n^\lambda(z) = 2\lambda C_{n-1}^{\lambda+1}(z),$$

$$\left[(1-z^2)\frac{d}{dz} - 2z\lambda + z\right]C_n^\lambda(z) = \frac{(n+1)(n+2\lambda-1)}{2(1-\lambda)} C_{n+1}^{\lambda-1}(z), \tag{1.8'}$$

$$zC_n^\lambda(z) = \frac{n+1}{2(\lambda+n)} C_{n+1}^\lambda(z) + \frac{(2\lambda+n-1)}{2(\lambda+n)} C_{n-1}^\lambda(z), \tag{1.9'}$$

$$C_n^\lambda(z) = \frac{\lambda}{\lambda+n}\left(C_n^{\lambda+1}(z) - C_{n-2}^{\lambda+1}(z)\right), \tag{1.10'}$$

$$2(\lambda-1)(1-z^2)C_n^\lambda(z) = \frac{-(n+2)(n+1)}{2(\lambda+n)} C_{n+2}^{\lambda-1}(z) + \frac{(n+2\lambda-1)(n+2\lambda-2)}{2(\lambda+n)} C_n^{\lambda-1}(z), \tag{1.11'}$$

valid for nonintegral $\lambda \in \mathcal{C}$, $n = 0, 1, 2, \cdots$.

There is another useful model of the representations $\rho_0(\omega)$, $\rho_\mu(\omega)$ which can be constructed in terms of differential operators in three complex variables. This model (Model B) is closely related to the separation of variables method for solution of the wave equation in spherical coordinates and is determined by the operators

$$J^3 = t\frac{\partial}{\partial t}, \quad J^+ = -t\frac{\partial}{\partial z},$$
$$J^- = t^{-1}\left((1-z^2)\frac{\partial}{\partial z} - 2zt\frac{\partial}{\partial t}\right),$$
$$P^3 = \omega\left[z\frac{\partial}{\partial r} + \frac{(1-z^2)}{r}\frac{\partial}{\partial z} - \frac{zt}{r}\frac{\partial}{\partial t}\right],$$
$$P^+ = \omega t\left(\frac{\partial}{\partial r} - \frac{z}{r}\frac{\partial}{\partial z} - \frac{t}{r}\frac{\partial}{\partial t}\right),$$
$$P^- = \omega t^{-1}\left[(1-z^2)\frac{\partial}{\partial r} - \frac{z(1-z^2)}{r}\frac{\partial}{\partial z} + \frac{(z^2+1)}{r}t\frac{\partial}{\partial t}\right]. \tag{1.15}$$

Notice that the J operators in expressions (1.13) and (1.15) coincide. Thus, to finish the construction of Model B based on operators (1.15), we look for basis vectors $f_m^{(u)}[r, z, t]$ of the form

$$f_m^{(u)}[r, z, t] = Z^{(u)}(r) f_m^{(u)}(z, t), \qquad (1.16)$$

where the functions $f_m^{(u)}(z, t)$ are given by Eq. (1.14). A straightforward computation shows that the basis vectors (1.16) and infinitesimal operators (1.15) satisfy relations (1.8)–(1.12) if and only if the functions $Z^{(u)}(r)$ satisfy the recursion relations

$$\left(\frac{d}{dr} - \frac{u}{r}\right) Z^{(u)}(r) = Z^{(u+1)}(r),$$
$$\left(\frac{d}{dr} + \frac{u+1}{r}\right) Z^{(u)}(r) = Z^{(u-1)}(r), \qquad (1.17)$$

for all values of u such that both sides of these expressions are defined. The solutions of these recursion relations are well known to be cylindrical functions.[1] For simplicity we shall primarily restrict ourselves to the solutions

$$Z^{(u)}(r) = r^{-\frac{1}{2}} I_{u+\frac{1}{2}}(r),$$

where $I_{u+\frac{1}{2}}(r)$ is a modified Bessel function

$$I_\lambda(r) = \sum_{k=0}^{\infty} \frac{(z/2)^{\lambda+2k}}{k!\,\Gamma(\lambda + k + 1)}.$$

Thus the basis vectors for Model B become

$$f_m^{(u)}[r, z, t] = (u - m)!\,\Gamma(m + \tfrac{1}{2}) r^{-\frac{1}{2}} I_{u+\frac{1}{2}}(r) C_{u-m}^{m+\frac{1}{2}}(z)(2t)^m. \quad (1.18)$$

As before, in the case of the representation $\rho_0(\omega)$, m takes the values $u, u - 1, \cdots, -u$ and u runs over the nonnegative integers, while, in the case of the representation $\rho_\mu(\omega)$, m takes values u, $u-1$, $u-2, \cdots, u-\mu$ is an integer, $0 \le \operatorname{Re}\mu < 1$ and 2μ is not an integer. (Note that as far as special function theory is concerned, the above results are independent of ω. Hence, in the remainder of this paper, we shall always set $\omega = 1$.)

(2) Each of the representations $\rho_0(1)$, $\rho_\mu(1)$ of $\mathcal{T}_6$ induces a local representation of the Lie group T_6 defined by linear operators $\mathbf{T}(h)$, $h \in T_6$, acting on V.[6] These operators satisfy the group property $\mathbf{T}(h)\mathbf{T}(h') = \mathbf{T}(hh')$ for h and h' in a sufficiently small neighborhood of the identity. The general theory relating local representations of Lie groups to representations of Lie algebras will not be repeated here.[6] We shall limit ourselves to construction of the operators $\mathbf{T}(h)$ and computation of the matrix elements of these operators with respect to the basis $\{f_m^{(u)}\}$. The results when applied to Models A and B constructed in (1) yield addition theorems and other identities relating Gegenbauer polynomials and cylindrical functions.

2. COMPUTATIONAL IDENTITIES

In this section we collect together several computational results which will be needed later to extend the Lie algebra representations $\rho_0(1)$ and $\rho_\mu(1)$ ($\omega = 1$) to local group representations of T_6. Assume that the operators $J^\pm$, J^3, $P^\pm$, P^3 and the basis vectors $f_m^{(u)}$ satisfy relations (1.8)–(1.12) and that they define either of the irreducible representations $\rho_0(1)$ or $\rho_\mu(1)$. (Formally, the results for both representations look the same. The difference lies only in the allowable values of u and m.)

Lemma 1:

$$C_l^{m+\frac{1}{2}}(P^3) f_m^{(u)} = \sum_{k=0}^{\min(l,u-m)} A(m + \tfrac{1}{2}; l, u - m; k) f_m^{(u+l-2k)},$$

where

$$A(\lambda; l, s; k) = \frac{s!\,\Gamma(2\lambda + s + l - k)\Gamma(\lambda + l - k)\Gamma(\lambda + k)}{(s-k)!\,(l-k)!\,k!\,\Gamma(2\lambda + s + l - 2k)\,\Gamma(\lambda + s + l - k + 1)} \times \frac{\Gamma(\lambda + s - k)}{\Gamma^2(\lambda)}(\lambda + s + l - 2k),$$

if $0 \le k \le \min(l, s)$

$= 0$, otherwise.

Here, $\lambda \in \mathcal{C}$ and l, s, k are nonnegative integers.

Proof: Straightforward induction on l, using the recursion relations (1.9) and (1.9′).

Lemma 1 is a consequence merely of the abstract definition of the representations $\rho_0(1)$ and $\rho_\mu(1)$. Hence, the lemma must be valid for Models A and B. In Model A, $P^3 = z$ and $f_m^{(u)}$ is given by Eq. (1.14). We immediately obtain the known result:

Corollary 1:

$$C_l^\lambda(z) C_s^\lambda(z) = \sum_{k=0}^{\min(l,s)} A(\lambda; l, s; k) \frac{(s + l - 2k)!}{s!} \cdot C_{l+s-2k}^\lambda(z).$$

For Model B, we obtain

Corollary 2:

$$C_l^\lambda\left(z\frac{\partial}{\partial r} + \frac{(1 - z^2)}{r}\frac{\partial}{\partial z} - \frac{z(\lambda - \frac{1}{2})}{r}\right) \frac{I_{s+\lambda}(r)}{\sqrt{r}} C_s^\lambda(z) = \sum_{k=0}^{\min(l,s)} A(\lambda; l, s; k) \frac{(s + l - 2k)!}{s!} \times \frac{I_{l+s-2k+\lambda}(r)}{\sqrt{r}} C_{l+s-2k}^\lambda(z).$$

When $s = 0$, this expression simplifies to the identity

$$C_l^\lambda\left(z\frac{\partial}{\partial r} + \frac{(1-z^2)}{r}\frac{\partial}{\partial z} - \frac{z}{r}(\lambda - \tfrac{1}{2})\right)\frac{I_\lambda(r)}{\sqrt{r}} = \frac{I_{l+\lambda}(r)}{\sqrt{r}}C_l^\lambda(z).$$

Lemma 2: Let $\nu \in \mathcal{C}$ and l be a nonnegative integer. Then

$$C_l^\nu(P^3)f_u^{(u)} = \sum_{k=0}^{[l/2]} B(\nu, u + \tfrac{1}{2}; l, k)f_u^{(u+l-2k)},$$

where

$$B(\nu, \lambda; l, k) = \frac{(\lambda + l - 2k)\Gamma(\nu + l - k)\Gamma(\nu - \lambda + k)\Gamma(\lambda)}{(l-2k)!\,k!\,\Gamma(\nu)\Gamma(\nu-\lambda)\Gamma(\lambda + l - k + 1)}.$$

Proof: Induction of l using (1.9) and (1.9′).

In the remainder of this section, λ is any complex number not an integer, such that 2λ is not a negative integer.

Corollary 3: Let $\nu \in \mathcal{C}$. Then

$$C_l^\nu(z) = \sum_{k=0}^{[l/2]} B(\nu, \lambda; l, k)(l-2k)!\,C_{l-2k}^\lambda(z).$$

Corollary 4: Let $\nu, \lambda \in \mathcal{C}$. Then

$$C_l^\nu\left[z\frac{\partial}{\partial r} + \frac{(1-z^2)}{r}\frac{\partial}{\partial z} - \frac{z}{r}(\lambda - \tfrac{1}{2})\right]\frac{I_\lambda(r)}{\sqrt{r}} = \sum_{k=0}^{[l/2]} B(\nu, \lambda; l, k)(l-2k)!\,\frac{I_{\lambda+l-2k}(r)}{\sqrt{r}}C_{l-2k}^\lambda(z).$$

Lemma 3:

$$(P^3)^l f_u^{(u)} = \sum_{k=0}^{[l/2]}\frac{(u + l - 2k + \tfrac{1}{2})}{2^l} \times \frac{\Gamma(u+\tfrac{1}{2})l!}{\Gamma(u + l - k + \tfrac{3}{2})k!\,(l-2k)!}f_u^{(u+l-2k)}.$$

Proof: Relation (1.9) and induction on l.

Corollary 5:

$$\frac{(2z)^l}{l!} = \sum_{k=0}^{[l/2]}\frac{(\lambda + l - 2k)\Gamma(\lambda)}{\Gamma(\lambda + l - k + 1)k!}C_{l-2k}^\lambda(z).$$

Corollary 6:

$$\left[z\frac{\partial}{\partial r} + \frac{(1-z^2)}{r}\frac{\partial}{\partial z} - \frac{z}{r}(\lambda - \tfrac{1}{2})\right]^l\frac{I_\lambda(r)}{\sqrt{r}} = \sum_{k=0}^{[l/2]}\frac{(\lambda + l - 2k)\Gamma(\lambda)l!}{2^l\Gamma(\lambda + l - k + 1)k!}\frac{I_{\lambda+l-2k}(r)}{\sqrt{r}}C_{l-2k}^\lambda(z).$$

Lemma 4:

$$(P^+)^l f_m^{(u)} = \sum_{k=0}^{l}\binom{l}{k} \times \frac{(-1)^k(u-m)!\,\Gamma(u + \tfrac{1}{2} - k)(u + \tfrac{1}{2} + l - 2k)}{2\Gamma(u - m - 2k + 1)\Gamma(u + l - k + \tfrac{3}{2})} \times f_{m+l}^{(u+l-2k)}.$$

Proof: Relation (1.10) and induction on l.

Lemma 5:

$$(P^-)^l f_m^{(u)} = \sum_{k=0}^{l}\binom{l}{k} \times \frac{(-1)^{l+k}\Gamma(u + m + 1) \times \Gamma(u - k + \tfrac{1}{2})(u + l - 2k + \tfrac{1}{2})}{2^l\Gamma(u + m - 2k + 1)\Gamma(u + l - k + \tfrac{3}{2})} \times f_{m-l}^{(u+l-2k)}.$$

Proof: Relation (1.11) and induction on l.

We can use the above lemmas to compute the action of the operators $\exp(\alpha P^3)$, $\exp(\alpha P^+)$, and $\exp(\alpha P^-)$ on V. [If P is a linear operator on V and $\alpha_k \in \mathcal{C}$, we define $\exp(\alpha P)$ to be the formal sum $\sum_{k=0}^\infty (\alpha/k!)P^k$.] Although these results will be of only formal significance for the abstract representations $\rho_0(1)$ and $\rho_\mu(1)$, we will soon see that when applied to Models A and B they can be rigorously justified.

Lemma 6:

$$e^{\alpha P^3}f_u^{(u)} = \sum_{k=0}^\infty\frac{(u + k + \tfrac{1}{2})}{k!}\left(\frac{\alpha}{2}\right)^{\frac{1}{2}-u-} \times I_{u+k+\frac{1}{2}}(\alpha)\Gamma(u + \tfrac{1}{2})f_u^{(u+k)}.$$

Proof: This result follows directly from Lemma 3.

Assuming that Lemma 6 is valid when applied to Model A, we find:

Corollary 7: If $\alpha, \lambda \in \mathcal{C}$, then

$$e^{\alpha z} = \left(\frac{2}{\alpha}\right)^\lambda\Gamma(\lambda)\sum_{k=0}^\infty(\lambda + k)I_{\lambda+k}(\alpha)C_k^\lambda(z).$$

Corollary 8:

$$e^{\alpha P^3} = \left(\frac{2}{\alpha}\right)^\lambda\Gamma(\lambda)\sum_{k=0}^\infty(\lambda + k)I_{\lambda+k}(\alpha)C_k^\lambda(P^3).$$

3. DETERMINATION OF THE OPERATORS T(*h*)

The differential operators (1.13), which define Model A, satisfy the commutation relations of the Lie algebra $\mathcal{E}_6$. Hence, according to standard results in Lie theory,[8] these operators uniquely determine a

[8] H. W. Guggenheimer, *Differential Geometry* (McGraw-Hill Book Co., New York, 1963), Chap. 7.

local representation of T_6 by operators $\mathbf{T}(h)$, $h \in T_6$, acting on the space of analytic functions in two complex variables. The computation of $\mathbf{T}(h)$ is straightforward,[6,8] and we merely give the results. Due to the group multiplication law (1.3), we can write

$$\mathbf{T}(h) = \mathbf{T}(\mathbf{w}, g) = \mathbf{T}(\mathbf{w}; \mathbf{e})\mathbf{T}(\mathbf{0}; g),$$

where

$$h = \{\mathbf{w}, g\}, \quad \mathbf{w} = (\alpha, \beta, \gamma) \in \mathcal{C}^3,$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2).$$

If f is defined and analytic in a neighborhood of some point $(z, t) \in \mathcal{C}^2$ $(t \neq 0)$, then we have

$$\begin{aligned}[\mathbf{T}(\mathbf{w}; \mathbf{e})f](z, t) &= [\exp(\alpha P^+ + \beta P^- + \gamma P^3)f](z, t) \\ &= \exp[\alpha t + \beta(1 - z^2)t^{-1} + \gamma z] \\ &\quad \times f(z, t). \qquad (3.1)\end{aligned}$$

Furthermore,

$$[\exp \alpha J^3 f](z, t) = f(z, te^\alpha),$$

$$[\exp \alpha J^+ f](z, t) = f(z - \alpha t, t),$$

$$[\exp \alpha J^- f](z, t) = f\left(z + \frac{\alpha(1 - z^2)}{t}, \right. \qquad (3.2)$$

$$\left. t - 2\alpha z - \alpha^2 \frac{(1 - z^2)}{t}\right).$$

Combining these results, we obtain

$$\begin{aligned}&[\mathbf{T}(\mathbf{0}; g)f](z, t) \\ &= [\exp(-b/dJ^+)\exp(-cdJ^-)\exp(-2\ln dJ^3)f](z, t) \\ &= f\left(z(1 + 2bc) + abt + \frac{cd(z^2 - 1)}{t}, \right. \\ &\qquad \left. a^2t + 2acz + c^2\frac{(z^2 - 1)}{t}\right), \qquad (3.3)\end{aligned}$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1.$$

By construction, the $\mathbf{T}$ operators satisfy the group multiplication property

$$\mathbf{T}(hh')f = \mathbf{T}(h)[\mathbf{T}(h')f], \qquad (3.4)$$

whenever both sides of this expression are well defined.

In the same way, the differential operators (1.15), which define Model B, can be used to construct a local representation of T_6 by operators $\mathbf{T}(h)$ acting on the space of analytic functions in three complex variables. As before, we write $\mathbf{T}(h) = \mathbf{T}(\mathbf{w}; g) = \mathbf{T}(\mathbf{w}; \mathbf{e})\mathbf{T}(\mathbf{0}; g)$. Standard techniques in Lie theory[8] give

$$\begin{aligned}&[\exp \alpha P^+ f](r, z, t) \\ &= f\left(r\left(1 + \frac{2t\alpha}{r}\right)^{\frac{1}{2}}, z\left(1 + \frac{2t\alpha}{r}\right)^{-\frac{1}{2}}, t\left(1 + \frac{2t\alpha}{r}\right)^{-\frac{1}{2}}\right),\end{aligned}$$

$$\begin{aligned}&[\exp \beta P^- f](r, z, t) \\ &= f\left(r\left(1 + \frac{2\beta(1 - z^2)}{rt}\right)^{\frac{1}{2}}, z\left(1 + \frac{2\beta(1 - z^2)}{rt}\right)^{-\frac{1}{2}}, \right. \\ &\qquad \left.\left(t + \frac{2\beta}{r}\right)\left(1 + \frac{2\beta(1 - z^2)}{rt}\right)^{-\frac{1}{2}}\right),\end{aligned}$$

$$\begin{aligned}&[\exp \gamma P^3 f](r, z, t) \\ &= f\left(r\left(1 + \frac{\gamma^2}{r^2} + \frac{2\gamma z}{r}\right)^{\frac{1}{2}}, \left(z + \frac{\gamma}{r}\right)\left(1 + \frac{\gamma^2}{r^2} + \frac{2\gamma z}{r}\right)^{-\frac{1}{2}}, \right. \\ &\qquad \left. t\left(1 + \frac{\gamma^2}{r^2} + \frac{2\gamma z}{r}\right)^{-\frac{1}{2}}\right).\end{aligned}$$

Thus,

$$\begin{aligned}&[\mathbf{T}(\mathbf{w}, \mathbf{e})f](r, z, t) \\ &\quad = [\exp \alpha P^+ \exp \beta P^- \exp \gamma P^3 f](r, z, t) \\ &\quad = f[rQ, (z + \gamma/r)Q^{-1}, (t + 2\beta/r)Q^{-1}], \qquad (3.5)\end{aligned}$$

where

$$Q = \left[1 + \frac{2\beta(1 - z^2)}{rt} + \frac{2\alpha}{r}\left(t + \frac{2\beta}{r}\right) + \frac{\gamma^2}{r^2} + \frac{2\gamma z}{r}\right]^{\frac{1}{2}}. \qquad (3.6)$$

Here f is defined and analytic in some neighborhood of the point $(r, z, t) \in \mathcal{C}^3$. Exactly as in the computation (3.3) we find

$$\begin{aligned}&[\mathbf{T}(\mathbf{0}; g)f](r, z, t) \\ &= f\left(r, z(1 + 2bc) + abt + cd\frac{(z^2 - 1)}{t}, \right. \\ &\qquad \left. a^2t + 2acz + c^2\frac{(z^2 - 1)}{t}\right). \qquad (3.7)\end{aligned}$$

Again, we have the group multiplication property

$$\mathbf{T}(hh')f = \mathbf{T}(h)[\mathbf{T}(h')f],$$

whenever both sides of this expression are well defined as analytic functions of r, z, and t.

4. MATRIX ELEMENTS OF $\rho_0(1)$

We will now compute the matrix elements of the group representation of T_6 induced by the Lie algebra representation $\rho_0(1)$ of $\mathcal{T}_6$. The restriction of this group representation to the real subgroup E_6 of T_6 is well known (it is a member of the so-called principal series of representations of E_6) and the restricted matrix elements have been computed.[9–11] We carry out the computation for T_6 here to motivate the more complicated work to follow in the next section and also to point out the increased information about special functions obtained by studying the complex group.

In the remainder of this section, u and v will be nonnegative integers, while m and n will be integers ranging over values from $-u$ to u and $-v$

[9] N. Y. Vilenkin, E. L. Akim, and A. A. Levin, Dokl. Akad. Nauk SSSR **112**, 987 (1957).

[10] N. Y. Vilenkin, *Translations of the Moscow Mathematical Society for the Year 1963* (American Mathematical Society, Providence, 1965), English Transl., pp. 209–290.

[11] W. Miller, Commun. Pure Appl. Math. **17**, 527 (1964).

to v, respectively. We define the matrix elements $\{v, n \,|\mathbf{w}, g|\, u, m\}$ of the representation $\rho_0(1)$ by

$$\mathbf{T}(\mathbf{w}, g)f_m^{(u)} = \sum_{v=0}^{\infty} \sum_{n=-v}^{v} \{v, n|\, \mathbf{w}, g\,|u, m\} f_n^{(v)}, \quad (4.1)$$

where the operator $\mathbf{T}(\mathbf{w}, g)$ and the functions $f_m^{(u)}$ refer either to Model A or to Model B. It is known[12] that the functions $f_m^{(u)}$ for both Models A and B form an analytic basis for the representation space in the sense of Ref. 6, Chap. 2. In particular, the functions $\mathbf{T}(\mathbf{w}, g)f_m^{(u)}$ can be expressed uniquely as linear combination of basis functions uniformly convergent in suitable domains. The coefficients in this expansion are bounded linear functionals of the argument $\mathbf{T}(\mathbf{w}, g)f_m^{(u)}$ (in the topology of uniform convergence on compact sets). Since these conditions are satisfied, it can be shown that the matrix elements $\{v, n|\, \mathbf{w}, g\,|u, m\}$ are model-independent: They are determined uniquely by the infinitesimal operators (1.8)–(1.11) and are the same for every model of $\rho_0(1)$ for which the functions $f_m^{(u)}$ form an analytic basis.[6] Thus the matrix elements can be computed directly from expressions (1.8)–(1.11) and they will be valid for both Models A and B.

Furthermore, the group property

$$\mathbf{T}(\mathbf{w}, g)\mathbf{T}(\mathbf{w}', g') = \mathbf{T}(\mathbf{w} + g\mathbf{w}', gg')$$

leads immediately to the addition theorem

$$\sum_{v'=0}^{\infty} \sum_{n'=-v'}^{v'} \{v, n|\, \mathbf{w}, g\,|v', n'\}\{v', n'|\, \mathbf{w}', g'\,|u, m\} = \{v, n|\, \mathbf{w} + g\mathbf{w}', gg'\,|u, m\} \quad (4.2)$$

for the matrix elements.[6]

Matrix elements of the form $\{v, n|\, \mathbf{0}, g\,|u, m\}$ are determined completely by the J operators (1.8) and depend only on the representation theory of $SL(2)$. In fact, for fixed u, the functions $f_m^{(u)}$ form a basis for the $(2u + 1)$-dimensional irreducible representation of $sl(2)$. The matrix elements of these irreducible representations are well-known.[6] We quote the results:

$$\begin{aligned}\{v, n|\, \mathbf{0}, g\,|u, m\} &= \frac{d^{u-n}a^{u+m}b^{n-m}(u-m)!}{(u-n)!} \times \frac{F(n-u, -m-u; n-m+1; bc/ad)}{\Gamma(n-m+1)}\,\delta_{v,u} \\ &= \frac{d^{u-m}a^{u+n}c^{m-n}(u+m)!}{(u+n)!} \times \frac{F(m-u, -n-u; m-n+1; bc/ad)}{\Gamma(m-n+1)}\,\delta_{v,u}, \end{aligned} \quad (4.3)$$

where

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \quad ad - bc = 1.$$

These expressions make sense even when the gamma function in the denominator has a singularity, since

$$\begin{aligned}\lim_{c\to -n} \frac{F(a, b; c; x)}{\Gamma(c)} &= \frac{a(a+1)\cdots(a+n)b(b+1)\cdots(b+n)}{(n+1)!} \\ &\quad \times x^{n+1}F(a+n+1, b+n+1; n+2; x), \\ &\qquad n = 0, 1, 2, \cdots. \end{aligned} \quad (4.4)$$

The hypergeometric functions in Eq. (4.3) are Jacobi polynomials.

It follows immediately that the identity

$$\mathbf{T}(\mathbf{0}; g)f_m^{(u)} = \sum_{n=-u}^{u} \{u, n|\, \mathbf{0}, g\,|u, m\} f_n^{(u)} \quad (4.5)$$

must be valid for both Models A and B. Substituting expressions (1.14) and (3.3) for Model A into (4.5) and simplifying, we easily obtain the identity

$$\begin{aligned}&\frac{k!\,\Gamma(u-k+\frac{1}{2})}{(2u-k)!}\left(\frac{x^2}{2}\right)^k C_k^{u-k+\frac{1}{2}} \\ &\quad \times [z^2 - z - 1 + (2z-1)/x + 1/x^2] \\ &\quad \times (1 + 2xz + x^2(z^2-1))^{u-k} \\ &= \sum_{l=0}^{2u} \frac{l!}{(2u-l)!}\Gamma(u-l+\tfrac{1}{2})\left(\frac{x}{2}\right)^l \\ &\quad \times \frac{F(-k, -2u+l; l-k+1; 1-x)}{\Gamma(l-k+1)} C_l^{u-l+\frac{1}{2}}(z). \end{aligned} \quad (4.6)$$

When $k = 0$, this identity reduces to a simple generating function

$$[1 + 2xz + x^2(z^2-1)]^u = \sum_{l=0}^{2u} \binom{2u}{l}\binom{u-\frac{1}{2}}{l}^{-1}\left(\frac{x}{2}\right)^l C_l^{u-l+\frac{1}{2}}(z)$$

for the basis vectors (1.14). Model B gives no new results.

Combining Lemma 1 and Corollary 8 we find

$$\begin{aligned}\mathbf{T}(0, 0, \gamma; \mathbf{e})f_m^{(u)} &= \exp(\gamma P^3) f_m^{(u)} \\ &= \left(\frac{2}{\gamma}\right)^{m+\frac{1}{2}} \Gamma(m+\tfrac{1}{2}) \sum_{l=0}^{\infty} (m+l+\tfrac{1}{2}) \\ &\quad \times I_{m+l+\frac{1}{2}}(\gamma) C_l^{m+\frac{1}{2}}(P^3) f_m^{(u)} \\ &= \left(\frac{2}{\gamma}\right)^{m+\frac{1}{2}} \Gamma(m+\tfrac{1}{2}) \sum_{j=-\infty}^{\infty} f_m^{(u+j)} \\ &\quad \times \sum_{k=0}^{\infty} A(m+\tfrac{1}{2}; j+2k, u-m; k) \\ &\quad \times (m+j+2k+\tfrac{1}{2}\; I_{m+j+2k+\frac{1}{2}}(\gamma). \end{aligned}$$

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[12] F. W. Schäfke, *Einführung in die Theorie der Speziellen Funktionen der Mathematischen Physik* (Springer-Verlag, Berlin, 1963), Chap. 8.

Therefore,

$$\begin{aligned}\{v, n \mid o, o, \gamma; \mathbf{e} \mid u, m\} &= \delta_{n,m}(2/\gamma)^{m+\frac{1}{2}}\Gamma(m+\tfrac{1}{2}) \\ &\quad \times \sum_{k=0}^{\infty} A(m+\tfrac{1}{2}; v-u+2k, u-m; k) \\ &\quad \times (m+v-u+2k+\tfrac{1}{2}) I_{m+v-u+2k+\frac{1}{2}}(\gamma). \quad (4.7)\end{aligned}$$

[Due to the properties of the symbol $A(\ )$, defined by Lemma 1, this sum is actually finite.] When $m = u$, we have the special case

$$\begin{aligned}\{v, n \mid 0, 0, \gamma; \mathbf{e} \mid u, u\} &= \delta_{n,u}(2/\gamma)^{u+\frac{1}{2}} \frac{\Gamma(u+\frac{1}{2})(v+\frac{1}{2})}{(v-u)!} I_{v+\frac{1}{2}}(\gamma), \quad \text{if} \quad v \geq u; \\ &= 0, \quad \text{if} \quad v < u, \quad (4.8)\end{aligned}$$

which also follows directly from Lemma 6.

The matrix element

$$\{u, m \mid \alpha, \beta, \gamma; \mathbf{e} \mid 0, 0\}$$

can be computed by making use of the identity

$$\begin{aligned}\{u, m \mid ab\xi, -cd\xi, (1+2bc)\xi; \mathbf{e} \mid 0, 0\} \\ = \{u, m \mid \mathbf{0}; g \mid u, 0\}\{u, 0 \mid 0, 0, \xi; \mathbf{e} \mid 0, 0\}, \quad (4.9)\end{aligned}$$

where $g \in SL(2)$. This identity is a special case of the addition theorem (4.2). In terms of new variables $\alpha = ab\xi$, $\beta = -cd\xi$, $\gamma = (1+2bc)\xi$, and $\rho^2 = \gamma^2 + 4\alpha\beta$, the matrix elements on the right-hand side of Eq. (4.9) are given by

$$\begin{aligned}\{u, m \mid \mathbf{0}; g \mid u, 0\} &= \frac{\Gamma(|m|+\frac{1}{2})u!}{\sqrt{\pi}(u+|m|)!}\left(\frac{4}{\rho}\right)^{|m|} \alpha^{(|m|+m)/2}(-\beta)^{(|m|-m)/2} \\ &\qquad \times C_{u-|m|}^{|m|+\frac{1}{2}}(\gamma/\rho), \\ \{u, 0 \mid 0, 0, \xi; \mathbf{e} \mid 0, 0\} &= (2/\rho)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2})}{u!}(u+\tfrac{1}{2}) I_{u+\frac{1}{2}}(\xi). \quad (4.10)\end{aligned}$$

Therefore,

$$\begin{aligned}\{u, m \mid \alpha, \beta, \gamma; \mathbf{e} \mid 0, 0\} &= (2/\rho)^{\frac{1}{2}}(4/\rho)^{|m|}\alpha^{(|m|+m)/2}(-\beta)^{(|m|-m)/2} \\ &\quad \times \frac{\Gamma(|m|+\frac{1}{2})(u+\frac{1}{2})}{(u+|m|)!} C_{u-|m|}^{|m|+\frac{1}{2}}(\gamma/\rho) I_{u+\frac{1}{2}}(\rho). \quad (4.11)\end{aligned}$$

There is an ambiguity in the signs of expressions (4.10) since $\rho = \pm[\gamma^2 + 4\alpha\beta]^{\frac{1}{2}}$. However, a close inspection of (4.11) reveals that the final matrix element is a function of ρ^2 so the ambiguity in sign causes no harm. Furthermore, the matrix element is an entire function of α, β, and γ.

Applying the identity

$$\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e}) f_0^{(0)} = \sum_{u=0}^{\infty} \sum_{m=-u}^{u} \{u, m \mid \alpha, \beta, \gamma; \mathbf{e} \mid 0, 0\} f_m^{(u)}$$

to Model A, we obtain

$$\begin{aligned}\exp[\alpha t + \beta(1-z^2)/t + \gamma z] &= \left(\frac{2}{\pi\rho}\right)^{\frac{1}{2}} \sum_{u=0}^{\infty} \sum_{m=0}^{u} (8t\alpha/\rho)^m (u+\tfrac{1}{2})(u-m)! \frac{\Gamma^2(m+\frac{1}{2})}{(u+m)!} \\ &\quad \times I_{u+\frac{1}{2}}(\rho) C_{u-m}^{m+\frac{1}{2}}(\gamma/\rho) C_{u-m}^{m+\frac{1}{2}}(z) \\ &\quad + \left(\frac{2\pi}{\rho}\right)^{\frac{1}{2}} \sum_{u=0}^{\infty} \sum_{m=0}^{u} (2\beta/t\rho)^m (u+\tfrac{1}{2}) \\ &\quad \times I_{u+\frac{1}{2}}(\rho) C_{u-m}^{m+\frac{1}{2}}(\gamma/\rho) C_{u+m}^{-m+\frac{1}{2}}(z). \quad (4.12)\end{aligned}$$

This formula is the complex generalization of the well-known formula

$$e^{i\mathbf{p}\cdot\mathbf{r}} = \left(\frac{8\pi^3}{pr}\right)^{\frac{1}{2}} \sum_{l=0}^{\infty} \sum_{k=-l}^{l} i^l J_{l+\frac{1}{2}}(pr) Y_l^k(\Theta_r, \varphi_r) Y_l^k(\Theta_p, \varphi_p)$$

for the expansion of plane waves into spherical waves. Since the left-hand side of Eq. (4.12) is an entire function of the variables αt, β/t, γ, and z, it follows from standard expansion theorems for Gegenbauer polynomials[12] that the right-hand side must converge for all values of these variables. Furthermore, the expansion coefficients $\{u, m \mid \alpha, \beta, \gamma; \mathbf{e} \mid 0, 0\}$ on the right-hand side must be entire functions of α, β, and γ.

At this point, we can fill a gap in our derivation of Eq. (4.11). This derivation was valid only for $\rho \neq 0$. However, using Model A, we have seen that the required matrix element is an entire function of α, β, and γ. Thus to compute $\{u, m \mid \alpha, \beta, \gamma; \mathbf{e} \mid 0, 0\}$ for $\gamma^2 + 4\alpha\beta = 0$ we need only find the value of Eq. (4.11) as $\rho \to 0$. The result is

$$\begin{aligned}\{u, m \mid \alpha, \beta, \gamma; \mathbf{e} \mid 0, 0\} &= \frac{(2\alpha)^m \gamma^{u-m}}{(u+m)!(u-m)!}, \quad \text{if} \quad m \geq 0, \quad \rho = 0, \\ &= \frac{(-2\beta)^{-m}\gamma^{u+m}}{(u-m)!(u+m)!}, \quad \text{if} \quad m \leq 0, \quad \rho = 0. \quad (4.13)\end{aligned}$$

We are now in a position to calculate the general matrix element $\{v, n \mid \alpha, \beta, \gamma; \mathbf{e} \mid u, m\}$. Using Model A, we find

$$\begin{aligned}\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e}) f_m^{(u)} &= (u-m)!\,\Gamma(m+\tfrac{1}{2})(2t)^m \\ &\quad \times \exp[\alpha t + (1-z^2)\beta/t + \gamma z] C_{u-m}^{m+\frac{1}{2}}(z) \\ &= (u-m)!\,\Gamma(m+\tfrac{1}{2}) \sum_{r=0}^{\infty} \sum_{k=-r}^{r} \{r, k \mid \alpha, \beta, \gamma; \mathbf{e} \mid 0, 0\} \\ &\quad \times C_{u-m}^{m+\frac{1}{2}}(z) C_{r-k}^{k+\frac{1}{2}}(z)(2t)^{m+k}(r-k)!\,\Gamma(k+\tfrac{1}{2}).\end{aligned}$$

From the connection between Gegenbauer polynomials and the representation theory of $SL(2)$, it follows that

$$\begin{aligned}(u-m)!\,(r-k)!\,\Gamma(m+\tfrac{1}{2})\Gamma(k+\tfrac{1}{2})C_{u-m}^{m+\frac{1}{2}}(z)C_{r-k}^{k+\frac{1}{2}}(z)\\ = [\pi(u-m)!\,(r-k)!\,(u+m)!\,(r+k)!]^{\frac{1}{2}}\\ \times \Gamma(m+k+\tfrac{1}{2}) \sum_{s=0}^{2\min(u,r)} \left[\frac{(u+r-s-m-k)!}{(u+r-s+m+k)!}\right]^{\frac{1}{2}}\\ \times C(u,0;r,0\,|\,u+r-s,0)\\ \times C(u,m;r,k\,|\,u+r-s,m+k)C_{u+r-s-m-k}^{m+k+\frac{1}{2}}(z),\end{aligned} \tag{4.14}$$

where the $C(.\,;.\,|\,.)$ are ordinary Clebsch–Gordan coefficients.[6,13]

Thus,

$$\mathbf{T}(\alpha,\beta,\gamma;\mathbf{e})f_m^{(u)} = \sum_{v=0}^{\infty}\sum_{n=-v}^{v}\{v,n\,|\,\alpha,\beta,\gamma;\mathbf{e}\,|u,m\}f_n^{(v)},$$

where

$$\begin{aligned}\{v,n\,|\,\alpha,\beta,\gamma,\mathbf{e}\,|u,m\}\\ =\sum_s\Bigg[\frac{\pi(u-m)!\,(u+m)!\,(v-u+s+n-m)!}{(v-n)!\,(v+n)!}\\ \times(v-u+s+m-n)!\Bigg]^{\frac{1}{2}}\\ \times C(u,0;v-u+s,0\,|\,v,0)\\ \times C(u,m;v-u+s,n-m\,|\,v,n)\\ \times\{v-u+s,n-m\,|\,\alpha,\beta,\gamma;\mathbf{e}\,|0,0\},\end{aligned} \tag{4.15}$$

and s ranges over the finite set of nonnegative integer values for which the summand is defined.

Now that all of the matrix elements of the representation $\rho_0(1)$ have been computed, it is a simple task to substitute these expressions into the addition theorem (4.2) and obtain identities for special functions. This will be left to the reader.

5. MATRIX ELEMENTS OF $\rho_\mu(1)$

The task of computing the matrix elements of the representation $\rho_\mu(1)$ is analogous to that for $\rho_0(1)$ but somewhat more complicated. In this section, u and v will be arbitrary complex numbers such that $2u$ and $2v$ are not integers and such that $u-v$ is an integer. The variables m, n will take values $m = u, u-1, u-2, \cdots; n = v, v-1, v-2, \cdots$.

As in Sec. 4, we define the matrix elements $\{v, n\,|\mathbf{w}; g|\,u, m\}$ of $\rho_\mu(1)$ by

$$\mathbf{T}(\mathbf{w};g)f_m^{(u)} = \sum_v\sum_n\{v,n\,|\,\mathbf{w};g\,|u,m\}f_n^{(v)}, \tag{5.1}$$

where the operator $\mathbf{T}(\mathbf{w}; g)$ and the basis functions $f_m^{(u)}$ refer either to Model A or Model B. Again, it follows that the functions $\{f_m^{(u)}\}$ for both Models A and B form an analytic basis for the representation space.[12] Thus the matrix elements are well defined and are uniquely determined by the infinitesimal operators (1.8)–(1.11).

[13] G. Y. Lyubarskii, *The Application of Group Theory to Physics* (Pergamon Press, Inc., Oxford, 1960), English Transl., Chap. 10.

Under the action of J^+, J^-, J^3, the vectors $\{f_m^{(u)}\}$ for fixed u, $m = u, u-1, u-2, \cdots$, form a basis for an irreducible representation of $sl(2)$. This representation, denoted by $\downarrow u$, was studied in Ref. 6, Chap. 5, and the matrix elements were computed to be

$$\begin{aligned}\{v,n\,|\,\mathbf{0},g\,|u,m\}\\ =\frac{d^{u-n}a^{u+m}b^{n-m}(u-m)!}{(u-n)!}\\ \times\frac{F(n-u,-m-u;n-m+1;bc/ad)}{\Gamma(n-m+1)}\delta_{v,u}\\ =\frac{d^{u-m}a^{u+n}c^{m-n}\Gamma(u+m+1)}{\Gamma(u+n+1)}\\ \times\frac{F(m-u,-n-u;m-n+1;bc/ad)}{\Gamma(m-n+1)}\delta_{v,u},\end{aligned} \tag{5.2}$$

where

$$g=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2),\quad ad-bc=1.$$

These matrix elements define a local representation of $SL(2)$: they are well defined and satisfy the group representation property only in a sufficiently small neighborhood of $\mathbf{e}$. Note, for example, in Eq. (5.2) that $(e^{2\pi i}a)^{u+m} \neq e^{2\pi i(u+m)}a^{u+m}$. A precise definition of this representation is worked out in Ref. 6 and will not be repeated here.

The identity

$$\mathbf{T}(\mathbf{0},g)f_m^{(u)} = \sum_{n\le u}\{u,n\,|\,\mathbf{0},g\,|u,m\}f_n^{(u)}$$

is valid for both Models A and B when g is in a sufficiently small neighborhood of $\mathbf{e}$. Substituting expressions (1.14) and (3.3) for Model A into this identity and simplifying, we obtain

$$\begin{aligned}\frac{k!\,\Gamma(u-k+\frac{1}{2})}{\Gamma(2u-k+1)}\left(\frac{x^2}{2}\right)^k\\ \times C_k^{u-k+\frac{1}{2}}[z^2-z-1+(2z-1)/x+1/x^2]\\ \times(1+2xz+x^2(z^2-1))^{u-k}\\ =\sum_{l=0}^{\infty}\frac{l!\,\Gamma(u-l+\frac{1}{2})}{\Gamma(2u-l+1)}\left(\frac{x}{2}\right)^l\\ \times\frac{F(-k,-2u+l;l-k+1;1-x)}{\Gamma(l-k+1)}C_l^{u-l+\frac{1}{2}}(z),\\ |2xz+x^2(z^2-1)|<1.\end{aligned} \tag{5.3}$$

The computation of the matrix element

$$\{v,n\,|0,0,\gamma;\mathbf{e}|\,u,m\}$$

is carried out exactly as for the corresponding element

(4.7) of $\rho_0(1)$:

$$\{v, n | 0, 0, \gamma; \mathbf{e} | u, m\} = \delta_{n,m} I_m^{v,u}(\gamma) = \delta_{n,m}(2/\gamma)^{m+\frac{1}{2}} \frac{(u-m)!\,(v+\frac{1}{2})}{\Gamma(m+\frac{1}{2})\Gamma(v+m+1)}$$
$$\times \sum_{k=0}^{\infty} \frac{(m+v-u+2k+\frac{1}{2})\Gamma(v+m+k+1)\Gamma(v-u+m+k+\frac{1}{2})}{(u-m-k)!\,(v-u+k)!\,k!\,\Gamma(v+k+\frac{3}{2})}$$
$$\times \Gamma(m+k+\tfrac{1}{2})\Gamma(u-k+\tfrac{1}{2}) I_{m+v-u+2k+\frac{1}{2}}(\gamma). \tag{5.4}$$

The difference is solely the domain of definition of u, v, m, and n. Note that the sum in Eq. (5.4) contains only a finite number of nonzero terms and that the matrix element is an entire function of γ.

The functions $I_m^{v,u}(\gamma)$ form a natural generalization of the ordinary modified Bessel function.[10] In fact, if $m = u$, we have

$$I_u^{v,u}(\gamma) = (2/\gamma)^{u+\frac{1}{2}} \frac{\Gamma(u+\frac{1}{2})(v+\frac{1}{2})}{(v-u)!} I_{v+\frac{1}{2}}(\gamma), \quad \text{if } v-u \geq 0,$$
$$= 0, \quad \text{if } v-u < 0. \tag{5.5}$$

The addition theorem

$$\{v, m | 0, 0, \gamma+\gamma'; \mathbf{e} | u, m\} = \sum_{k=-\infty}^{\infty} \{v, m | 0, 0, \gamma; \mathbf{e} | u+k, m\} \times \{u+k, m | 0, 0, \gamma'; \mathbf{e} | u, m\}$$

implies the identity

$$I_m^{v,u}(\gamma+\gamma') = \sum_{k=-\infty}^{\infty} I_m^{v,u+k}(\gamma) I_m^{u+k,u}(\gamma'). \tag{5.6}$$

Moreover, the identity

$$\mathbf{T}(0,0,\gamma;\mathbf{e}) f_m^{(u)} = \sum_{k=-\infty}^{\infty} \{u+k, m | 0, 0, \gamma; \mathbf{e} | u, m\} f_m^{(u+k)}$$

applied to Model A yields

$$l!\, e^{\gamma z} C_l^{m+\frac{1}{2}}(z) = \sum_{k=0}^{\infty} k!\, I_m^{m+k,m+l}(\gamma) C_k^{m+\frac{1}{2}}(z). \tag{5.7}$$

The right-hand side of this expression converges for all $\gamma, z \in \mathbb{C}$.

Using standard techniques from special function theory, we can apply relations (1.8′)–(1.10′) to the generating function (5.7) and derive recursion relations for the generalized Bessel functions. Among the results which can be obtained in this way are

$$\frac{(k+1)}{\gamma} I_m^{m+k+1,m+l}(\gamma) = \frac{1}{2m+2k+1} I_m^{m+k,m+l}(\gamma) - \frac{(k+1)(k+2)}{2m+2k+5} \times I_m^{m+k+2,m+l}(\gamma) + \frac{l}{\gamma} I_{m+1}^{m+k+1,m+l}(\gamma),$$

$$\frac{d}{d\gamma} I_m^{m+k,m+l}(\gamma) = \frac{1}{2m+2l+1} I_m^{m+k,m+l+1}(\gamma) + \frac{l(2m+l)}{2m+2l+1} I_m^{m+k,m+l-1}(\gamma)$$
$$= \frac{1}{2m+2k-1} I_m^{m+k-1,m+l}(\gamma) + \frac{(k+1)(2m+k+1)}{2m+2k+3} I_m^{m+k+1,m+l}(\gamma), \quad k, l = 0, 1, 2, \cdots.$$

Rather than compute directly an expression for the general matrix element $\{v, n | \alpha, \beta, \gamma; \mathbf{e} | u, m\}$ of $\rho_\mu(1)$, we will derive this result indirectly by determining a relation between the matrix elements of two different representations $\rho_\mu(1)$ and $\rho_{\mu'}(1)$. Denote the matrix elements of $\rho_{\mu'}(1)$ by $\{v', n' | \alpha, \beta, \gamma; \mathbf{e} | u', m'\}'$ to distinguish them from those of $\rho_\mu(1)$. (Our results will be valid even if $\mu' = 0$ or $\mu = 0$.)

Using Model A and Corollary 3, we find

$$f_m^{(u)}(z,t) = (2t)^{m-m'} \frac{(u-m)!}{\Gamma(m-m')} \times \sum_{k=0}^{[(u-m)/2]} \frac{(u+m'-m-2k+\frac{1}{2})}{k!\,(u-m-2k)!} \times \frac{\Gamma(u-k+\frac{1}{2})\Gamma(m-m'+k)}{\Gamma(m'-m+u-k+\frac{3}{2})} f_{m'}^{(m'+u-m-2k)}(z,t)$$
$$= (2t)^{m-m'} \sum_k D(u, m, m', k) f_{m'}^{(m'+u-m-2k)}(z,t),$$

where the basis functions $f_m^{(u)}(z, t)$ are given by Eq. (1.14). Applying the operator

$$\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e}) = \exp\,[\alpha t + \beta(1-z^2)/t + \gamma z]$$

to both sides of this equation and using Eq. (5.1) to expand each side in terms of its corresponding basis functions, we obtain the identity

$$\sum_{v,n} \{v, n | \alpha, \beta, \gamma; \mathbf{e} | u, m\} f_n^{(v)}(z,t) = (2t)^{m-m'} \times \sum_{k,v',n'} \{v', n' | \alpha, \beta, \gamma; \mathbf{e} | m'+u-m-2k, m'\}' \times D(u, m, m', k) f_{n'}^{(v')}(z,t).$$

Finally, using Corollary 3, again, to express the functions $f_n^{(v')}(z,t)$ as linear combinations of functions $f_n^{(v)}(z,t)$, $(n = m - m' + n')$, and equating coefficients of $f_n^{(v)}(z,t)$ on both sides of the identity, we derive the equality

$$\begin{aligned}\{v, n| \alpha, \beta, \gamma; \mathbf{e}|u, m\} &= \frac{(u-m)!\,(v+\frac{1}{2})}{\Gamma(m-m')\Gamma(m'-m)(v-n)!} \\ &\times \sum_{k=0}^{[(u-m)/2]} \sum_s \frac{(m'+u-m-2k+\frac{1}{2})(v-n+2s)!\,\Gamma(u-k+\frac{1}{2})}{k!\,s!\,(u-m-2k)!\,\Gamma(m'+u-m-k+\frac{3}{2})} \\ &\times \frac{\Gamma(m-m'+k)\Gamma(m'+v-m+s+\frac{1}{2})\Gamma(m'-m+s)}{\Gamma(v+s+\frac{3}{2})} \\ &\times \{m'+v-m+2s, m'+n-m| \alpha, \beta, \gamma; \mathbf{e}|m'+u-m-2k, m'\}'. \qquad (5.8)\end{aligned}$$

Here s ranges over all nonnegative integral values such that the summand is well defined.

Formula (5.8) can be employed to evaluate the matrix elements of $\rho_\mu(1)$. For example, set $m = u$, $m' = 0$, and use expression (4.11) for the primed elements on the right-hand side of Eq. (5.8). The result is

$$\begin{aligned}\{v, n| \alpha, \beta, \gamma; \mathbf{e}|u, u\} &= \left(\frac{2}{\rho\pi}\right)^{\frac{1}{2}} \left(\frac{4}{\rho}\right)^{|n-u|} \frac{(v+\frac{1}{2})\Gamma(u+\frac{1}{2})}{(v-m)\Gamma(-u)} \\ &\times \Gamma(|n-u|+\tfrac{1}{2})\alpha^{(|n-u|+u-n)/2}(-\beta)^{(|n-u|+u-n)/2} \\ &\times \sum_s \frac{(v-u+2s+\frac{1}{2})(v-n+2s)!\,\Gamma(v-u+s+\frac{1}{2})\Gamma(-u+s)}{s!\,\Gamma(v+s+\frac{3}{2})(|n-u|+v-u+2s)!} \\ &\times C^{|n-u|+\frac{1}{2}}_{v-u-|n-u|+2s}(\gamma/\rho) I_{v-u+2s+\frac{1}{2}}(\rho), \quad \rho^2 = \gamma^2 + 4\alpha\beta. \qquad (5.9)\end{aligned}$$

For the case $\alpha = \beta = 0$, $n = m = u$, Eqs. (5.8) and (5.5) yield

$$\begin{aligned}&(\gamma/2)^{\nu-\lambda} I_\lambda(\gamma) \\ &= \sum_{s=0}^{\infty} \frac{\Gamma(\nu+s)\Gamma(\nu-\lambda+s)(\nu+2s)}{s!\,\Gamma(\nu-\lambda)\Gamma(\lambda+s+1)} I_{\nu+2s}(\gamma), \quad \lambda, \nu \in \mathbb{C}. \qquad (5.10)\end{aligned}$$

In addition to the general result (5.8), we list two special classes of matrix elements whose forms follow immediately from Lemmas 4 and 5:

$$\begin{aligned}&\{v, n| \alpha, 0, 0; \mathbf{e}|u, m\} \\ &= \frac{(\alpha/2)^{n-m}}{(v-n)!} \frac{(u-m)!}{\left(\dfrac{u-m+n-v}{2}\right)!} \\ &\times \frac{(-1)^{(u-m+n-v)/2}\Gamma\left(\dfrac{1+m-n+v+u}{2}\right)(v+\frac{1}{2})}{\left(\dfrac{v+n-u-m}{2}\right)!\,\Gamma\left(\dfrac{-m+u+n+v+3}{2}\right)},\end{aligned}$$

if $n - m - |v-u|$ is a nonnegative even integer,

$=0$, otherwise. (5.11)

$$\begin{aligned}&\{v, n| 0, \beta, 0; \mathbf{e}|u, m\} \\ &= \left(\frac{\beta}{2}\right)^{m-n} \frac{(-1)^{(m-n-u+v)/2}}{\left(\dfrac{m-n+u-v}{2}\right)!\left(\dfrac{m-n-u+v}{2}\right)!} \\ &\times \frac{\Gamma(u+m+1)\Gamma\left(\dfrac{n-m+u+v+1}{2}\right)(v+\frac{1}{2})}{\Gamma(v+n+1)\Gamma\left(\dfrac{u+m-n+v+3}{2}\right)},\end{aligned}$$

if $m - n - |u-v|$ is a nonnegative even integer,

$= 0$, otherwise. (5.12)

By construction, the matrix elements of $\rho_\mu(1)$ satisfy the addition theorem:

$$\begin{aligned}&\{v, n| \mathbf{w} + g\mathbf{w}'; gg'|u, m\} \\ &= \sum_{k=-\infty}^{\infty} \sum_{l=0}^{\infty} \{v, n| \mathbf{w}, g|u+k, u+k-l\} \\ &\times \{u+k, u+k-l| \mathbf{w}', g'|u, m\}, \qquad (5.13)\end{aligned}$$

for all $\mathbf{w}, \mathbf{w}' \in \mathbb{C}^3$ and for g, g' in a sufficiently small neighborhood of $\mathbf{e} \in SL(2)$. (In any given example the restriction on g and g' can usually be determined by inspection.) We will list a few special cases of this theorem.

When $m = u$, $g = g' = \mathbf{e}$, $\mathbf{w} = (\alpha, \beta, \gamma)$, and $\mathbf{w}' = (\alpha', o, o)$, relation (5.13) simplifies to

$$\{v, n|\, \alpha + \alpha', \beta, \gamma; \mathbf{e}\,|u, u\} = \sum_{k=0}^{\infty}\left(\frac{\alpha'}{2}\right)^k \frac{\Gamma(u+\frac{1}{2})}{k!\,\Gamma(u+k+\frac{1}{2})} \times \{v, n|\, \alpha, \beta, \gamma; \mathbf{e}\,|u+k, u+k\}, \tag{5.14}$$

where the matrix elements on both sides of this expression are defined by Eq. (5.9).

The relation

$$\{v, n|\, ab\rho, -cd\rho, (1+2bc)\rho; \mathbf{e}\,|u, m\} = \sum_s \{v, n|\, \mathbf{0}, g\,|v, s\}\{v, s|\, 0, 0, \rho; \mathbf{e}\,|u, s\} \times \{u, s|\, \mathbf{0}, g^{-1}\,|u, m\}$$

leads to the identity

$$\{v, n|\, \alpha, \beta, \gamma; \mathbf{e}\,|u, m\} = \sum_{k=0}^{\infty}\left(\frac{1+\gamma/\rho}{2}\right)^{2u+v-n-k}\left(\frac{1-\gamma/\rho}{2}\right)^{m-u+k}(\alpha/\rho)^{n-m} \times \frac{\Gamma(u+m+1)(v-u+k)!}{\Gamma(2u-k+1)(v-n)!} \times \frac{F\left(n-v, -u-v+k; n-u+k; \dfrac{\gamma-\rho}{\gamma+\rho}\right)}{\Gamma(n-u+k+1)} \times \frac{F\left(m-u, -2u+k; m-u+k+1; \dfrac{\gamma-\rho}{\gamma+\rho}\right)}{\Gamma(m-u+k+1)} \times I_{u-k}^{v,u}(\rho), \tag{5.15}$$

valid for $|1 - z/\rho| < 2$. Here, $\rho = z[1 + 4xy/z^2]^{\frac{1}{2}}$. In case $\gamma = 0$, the identity becomes

$$\{v, n|\, \alpha, \beta, 0; \mathbf{e}\,|u, m\} = \sum_{k=0}^{\infty}(\tfrac{1}{2})^{u+v}(\alpha/\beta)^{(n-m)/2}\Gamma(u+m+1)(v-u+k)! \times \frac{F(m-u, -2u+k; m-u+k+1; -1) \times F(n-v, -u-v+k; n-u+k; -1)}{\Gamma(2u-k+1)(v-n)!\,\Gamma(m-u+k+1) \times \Gamma(n-u+k+1)} \times I_{u-k}^{v,u}(2\sqrt{\alpha\beta}), \quad \text{if } v+n-u-m \text{ is even},$$

$= 0$ otherwise.

Finally, we note the result

$$\{v, n|\, g\boldsymbol{\gamma}; g\,|u, m\} = \{v, n|\, \mathbf{0}, g\,|v, m\}\{v, m|\, \boldsymbol{\gamma}, \mathbf{e}\,|u, m\} = \sum_s \{v, n|\, g\boldsymbol{\gamma}; \mathbf{e}\,|u, s\}\{u, s|\, \mathbf{0}; g\,|u, m\},$$

where

$$\boldsymbol{\gamma} = (0, 0, \gamma), \quad g\boldsymbol{\gamma} = [ab\gamma, -cd\gamma, (1+2bc)\gamma],$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \rho = z(1 + 4xy/z^2)^{\frac{1}{2}}.$$

If $u \geq n$ and $u = m$, this implies

$$\frac{\Gamma(u+v+1)\Gamma(u+\frac{1}{2})(v+\frac{1}{2})}{\Gamma(v+n+1)(v-u)!\,(u-n)!} \times \left(\frac{z+\rho}{2\rho}\right)^{v-u}(-y)^{u-n}\left(\frac{2}{\rho}\right)^{u+\frac{1}{2}} \times F\left(u-v, -n-v; u-n+1; \frac{z-\rho}{z+\rho}\right) I_{v+\frac{1}{2}}(\rho) = \sum_{k=0}^{\infty}\{v, n|\, \alpha, \beta, \gamma; \mathbf{e}\,|u, u-k\} \times \left(\frac{z+\rho}{2}\right)^{u-n-k}\frac{(-y)^k}{k!}\,\frac{\Gamma(2u+1)}{\Gamma(2u-k+1)}.$$

There is a similar result for $n \geq u$.

6. APPLICATIONS TO MODEL B

Now that we have succeeded in computing matrix elements of the representations $\rho_0(1)$ and $\rho_\mu(1)$ we can apply our results to any model of these representations and obtain identities for special functions. As an illustration, consider Model B.

According to the work of Sec. 1, the basis vectors for Model B take the form

$$f_m^{(u)}[r, z, t] = Z^{(u)}(r)(u-m)!\,\Gamma(m+\tfrac{1}{2})C_{u-m}^{m+\frac{1}{2}}(z)(2t)^m,$$

where the $Z^{(u)}(r)$ satisfy recursion relations (1.17). Both the functions

$$Z^{(u)}(r) = r^{-\frac{1}{2}}I_{u+\frac{1}{2}}(r) \quad \text{and} \quad Z^{(u)}(r) = r^{-\frac{1}{2}}I_{-u-\frac{1}{2}}(r) \tag{6.1}$$

separately satisfy Eq. (1.17). Similarly, any linear combination of these functions satisfies Eq. (1.17). For purposes of illustration, we will use only the first of solutions (6.1). Recall that corresponding to the representation $\rho_0(1)$: $u = 0, 1, 2, \cdots$; $m = u, u-1, \cdots, -u$; while corresponding to $\rho_\mu(1)$: $u = \mu + k$; $k = 0, \pm 1, \pm 2, \cdots$; $m = u, u-1, u-2, \cdots$; $0 \leq \operatorname{Re}\mu < 1$ and 2μ is not an integer.

Since the functions $f_m^{(u)}[r, z, t]$ form an analytic basis for the representation space, we have immediately

$$[\mathbf{T}(\mathbf{w}; g)f_m^{(u)}][r, z, t] = \sum_{v,n}\{v, n|\, \mathbf{w}; g\,|u, m\}f_n^{(v)}[r, z, t], \tag{6.2}$$

where the operators $\mathbf{T}(\mathbf{w}; g)$ are given by Eqs. (3.5)–(3.7) and the matrix elements $\{v, n|\, \mathbf{w}; g\,|u, m\}$ are those computed in Secs. 4 and 5. The operators $\mathbf{T}(\mathbf{0}, g)$ yield no information which could not have been obtained from Model A. Therefore, we restrict ourselves to operators $\mathbf{T}(\mathbf{w}, \mathbf{e})$. In this case, Eq. (6.2) yields

$$
\begin{aligned}
&(u-m)!\,\Gamma(m+\tfrac{1}{2})I_{u+\frac{1}{2}}(rQ)\\
&\qquad\times C_{u-m}^{m+\frac{1}{2}}((z+\gamma/r)Q^{-1})Q^{-m-\frac{1}{2}}[2(t+2\beta/r)]^{m}\\
&\quad=\sum_{v,n}\{v,n\,|\,\mathbf{w};\mathbf{e}\,|u,m\}(v-n)!\,\Gamma(n+\tfrac{1}{2})\\
&\qquad\times I_{v+\frac{1}{2}}(r)C_{v-n}^{n+\frac{1}{2}}(z)(2t)^{n},
\end{aligned}
\tag{6.3}
$$

where

$$
Q=\left[1+\frac{2\beta(1-z)^{2}}{rt}+\frac{2\alpha}{r}\left(t+\frac{2\beta}{r}\right)+\frac{\gamma^{2}}{r^{2}}+\frac{2\gamma z}{r}\right]^{\frac{1}{2}}.
$$

When applied to the representation $\rho_0(1)$, Eq. (6.3) constitutes a generalization of the so-called addition theorem for spherical waves.[14] We will list a few special cases of Eq. (6.3), treating the representations $\rho_0(1)$ and $\rho_\mu(1)$ simultaneously.

If $\alpha=\beta=0$, Eq. (6.3) yields

$$
\begin{aligned}
&(u-m)!\,I_{u+\frac{1}{2}}(rR)C_{u-m}^{m+\frac{1}{2}}[(z+\gamma/r)R^{-1}]R^{-m-\frac{1}{2}}\\
&\quad=\sum_{k=m-u}^{\infty}(u+k-m)!\,I_{m}^{u+k,u}(\gamma)I_{u+k+\frac{1}{2}}(r)C_{u-m+k}^{m+\frac{1}{2}}(z),
\end{aligned}
\tag{6.4}
$$

where

$$
R=(1+2\gamma z/r+\gamma^{2}/r^{2})^{\frac{1}{2}},\quad |2\gamma z/r+\gamma^{2}/r^{2}|<1.
$$

When $m=u$, this expression simplifies to the well-known addition theorem of Gegenbauer:

$$
\begin{aligned}
&I_{u+\frac{1}{2}}(rR)(2R)^{-u-\frac{1}{2}}\\
&\quad=\Gamma(u+\tfrac{1}{2})\sum_{k=0}^{\infty}(u+k+\tfrac{1}{2})I_{u+k+\frac{1}{2}}(\gamma)I_{u+k+\frac{1}{2}}(r)C_{k}^{u+\frac{1}{2}}(z).
\end{aligned}
$$

[14] B. Friedman and J. Russek, Quart. Appl. Math. 12, 13 (1954).

There is an interesting special form of Eq. (6.4), obtained by setting $z=1$:

$$
\begin{aligned}
&(1+\gamma/r)^{-m-\frac{1}{2}}I_{u+\frac{1}{2}}(r+\gamma)\\
&\quad=\sum_{k=m-u}^{\infty}\frac{\Gamma(u+m+k+1)}{\Gamma(u+m+1)}I_{m}^{u+k,u}(\gamma)I_{u+k+\frac{1}{2}}(r),\\
&\qquad\qquad |\gamma/r|<1.
\end{aligned}
$$

When $m=u$, the above identity simplifies to

$$
\begin{aligned}
&(1+\gamma/r)^{-u-\frac{1}{2}}I_{u+\frac{1}{2}}(\cdot+\gamma)\\
&=(2/\gamma)^{u+\frac{1}{2}}\frac{\Gamma(u+\frac{1}{2})}{\Gamma(2u+1)}\\
&\quad\times\sum_{k=0}^{\infty}\frac{\Gamma(2u+k+1)(u+k+\frac{1}{2})}{k!}I_{u+k+\frac{1}{2}}(\gamma)I_{u+k+\frac{1}{2}}(r).
\end{aligned}
$$

If $\beta=\gamma=0$, Eqs. (6.3) and (5.11) give

$$
\begin{aligned}
&I_{u+\frac{1}{2}}(rS)C_{u-m}^{m+\frac{1}{2}}(zS^{-1})S^{-m-\frac{1}{2}}\\
&=\sum_{k=0}^{\infty}\sum_{j=0}^{[(u-m)/2]}\frac{(\alpha t)^{k}(-1)^{j}\Gamma(u-j+\frac{1}{2})(u+k-2j+\frac{1}{2})}{(u-m-2j)!\,j!\,(k-j)!\,\Gamma(u+k-j+\frac{3}{2})}\\
&\quad\times\frac{(u-m-2j)!\,\Gamma(m+k+\frac{1}{2})}{\Gamma(m+\frac{1}{2})}I_{u+k-2j+\frac{1}{2}}(r)\\
&\quad\times C_{u-m-2j}^{m+k+\frac{1}{2}}(z),
\end{aligned}
\tag{6.5}
$$

where

$$
S=(1+2\alpha t/r)^{\frac{1}{2}},\quad |2\alpha t/r|<1.
$$

When $m=u$, Eq. (6.5) reduces to

$$
I_{u+\frac{1}{2}}[r(1+2\alpha/r)^{\frac{1}{2}}](1+2\alpha/r)^{-u-\frac{1}{2}}=\sum_{k=0}^{\infty}\frac{\alpha^{k}}{k!}I_{u+k+\frac{1}{2}}(r),\quad |2\alpha/r|<1.
$$

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Special Functions and the Complex Euclidean Group in 3-Space. II

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This paper is the second in a series devoted to the derivation of identities for special functions which can be obtained from a study of the local irreducible representations of the Euclidean group in 3-space. A number of identities obeyed by Jacobi polynomials and Whittaker functions are derived and their group-theoretic meaning is discussed.

INTRODUCTION

Much of the theory of special functions, as it is applied in mathematical physics, is a disguised form of Lie group theory. The symmetry groups, which are built into the foundations of modern physics, determine many of the special functions which can arise in physics, as well as the principal properties of these functions. It is the author's opinion that a detailed analysis of this relationship between Lie theory and special functions is of importance for a good understanding of both special function theory and the laws of physics.

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$$(u-m)!\,\Gamma(m+\tfrac{1}{2})I_{u+\frac{1}{2}}(rQ) \times C^{m+\frac{1}{2}}_{u-m}((z+\gamma/r)Q^{-1})Q^{-m-\frac{1}{2}}[2(t+2\beta/r)]^m = \sum_{v,n}\{v,n|\,\mathbf{w};\mathbf{e}\,|u,m\}(v-n)!\,\Gamma(n+\tfrac{1}{2}) \times I_{v+\frac{1}{2}}(r)C^{n+\frac{1}{2}}_{v-n}(z)(2t)^n, \quad (6.3)$$

where

$$Q=\left[1+\frac{2\beta(1-z)^2}{rt}+\frac{2\alpha}{r}\left(t+\frac{2\beta}{r}\right)+\frac{\gamma^2}{r^2}+\frac{2\gamma z}{r}\right]^{\frac{1}{2}}.$$

When applied to the representation $\rho_0(1)$, Eq. (6.3) constitutes a generalization of the so-called addition theorem for spherical waves.[14] We will list a few special cases of Eq. (6.3), treating the representations $\rho_0(1)$ and $\rho_\mu(1)$ simultaneously.

If $\alpha=\beta=0$, Eq. (6.3) yields

$$(u-m)!\,I_{u+\frac{1}{2}}(rR)C^{m+\frac{1}{2}}_{u-m}[(z+\gamma/r)R^{-1}]R^{-m-\frac{1}{2}} = \sum_{k=m-u}^{\infty}(u+k-m)!\,I^{u+k,u}_m(\gamma)I_{u+k+\frac{1}{2}}(r)C^{m+\frac{1}{2}}_{u-m+k}(z), \quad (6.4)$$

where

$$R=(1+2\gamma z/r+\gamma^2/r^2)^{\frac{1}{2}},\quad |2\gamma z/r+\gamma^2/r^2|<1.$$

When $m=u$, this expression simplifies to the well-known addition theorem of Gegenbauer:

$$I_{u+\frac{1}{2}}(rR)(2R)^{-u-\frac{1}{2}} = \Gamma(u+\tfrac{1}{2})\sum_{k=0}^{\infty}(u+k+\tfrac{1}{2})I_{u+k+\frac{1}{2}}(\gamma)I_{u+k+\frac{1}{2}}(r)C^{u+\frac{1}{2}}_k(z).$$

[14] B. Friedman and J. Russek, Quart. Appl. Math. 12, 13 (1954).

There is an interesting special form of Eq. (6.4), obtained by setting $z=1$:

$$(1+\gamma/r)^{-m-\frac{1}{2}}I_{u+\frac{1}{2}}(r+\gamma) = \sum_{k=m-u}^{\infty}\frac{\Gamma(u+m+k+1)}{\Gamma(u+m+1)}I^{u+k,u}_m(\gamma)I_{u+k+\frac{1}{2}}(r),\quad |\gamma/r|<1.$$

When $m=u$, the above identity simplifies to

$$(1+\gamma/r)^{-u-\frac{1}{2}}I_{u+\frac{1}{2}}(\cdot+\gamma) = (2/\gamma)^{u+\frac{1}{2}}\frac{\Gamma(u+\frac{1}{2})}{\Gamma(2u+1)} \times \sum_{k=0}^{\infty}\frac{\Gamma(2u+k+1)(u+k+\frac{1}{2})}{k!}I_{u+k+\frac{1}{2}}(\gamma)I_{u+k+\frac{1}{2}}(r).$$

If $\beta=\gamma=0$, Eqs. (6.3) and (5.11) give

$$I_{u+\frac{1}{2}}(rS)C^{m+\frac{1}{2}}_{u-m}(zS^{-1})S^{-m-\frac{1}{2}} = \sum_{k=0}^{\infty}\sum_{j=0}^{[(u-m)/2]}\frac{(\alpha t)^k(-1)^j\Gamma(u-j+\frac{1}{2})(u+k-2j+\frac{1}{2})}{(u-m-2j)!\,j!\,(k-j)!\,\Gamma(u+k-j+\frac{3}{2})} \times\frac{(u-m-2j)!\,\Gamma(m+k+\frac{1}{2})}{\Gamma(m+\frac{1}{2})}I_{u+k-2j+\frac{1}{2}}(r) \times C^{m+k+\frac{1}{2}}_{u-m-2j}(z), \quad (6.5)$$

where

$$S=(1+2\alpha t/r)^{\frac{1}{2}},\quad |2\alpha t/r|<1.$$

When $m=u$, Eq. (6.5) reduces to

$$I_{u+\frac{1}{2}}[r(1+2\alpha/r)^{\frac{1}{2}}](1+2\alpha/r)^{-u-\frac{1}{2}} = \sum_{k=0}^{\infty}\frac{\alpha^k}{k!}I_{u+k+\frac{1}{2}}(r),\quad |2\alpha/r|<1.$$

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INTRODUCTION

Much of the theory of special functions, as it is applied in mathematical physics, is a disguised form of Lie group theory. The symmetry groups, which are built into the foundations of modern physics, determine many of the special functions which can arise in physics, as well as the principal properties of these functions. It is the author's opinion that a detailed analysis of this relationship between Lie theory and special functions is of importance for a good understanding of both special function theory and the laws of physics.

This paper is the second in a series analyzing the special function theory related to T_6, the complex Euclidean group in 3-space. In the first paper[1] (which we shall refer to as I), it was shown that an important class of identities relating Bessel functions and Gegenbauer polynomials had a simple interpretation in terms of certain local irreducible representations of T_6. In the present paper, which generalizes the results of I, a similar interpretation will be given for identities relating Whittaker functions and Jacobi polynomials.

Most of the identities for special functions derived in this paper are well known. We will be more interested in systematically deriving and uncovering the group-theoretic meaning of known identities than in the derivation of new identities.

Just as in I, the special functions obtained in this paper will arise in two ways: as matrix elements corresponding to local representations of T_6 and as basis vectors in a model of such a representation. Once the matrix elements of an abstract representation have been computed, they remain valid for any model of the representation. Only two models will be considered here, but the results of this paper can easily be extended to any other model which occurs in modern physical theories.

Finally, the reader will note that the algebraic and group-theoretic aspects of special function theory are emphasized at the expense of the analytic aspects. In particular, the order of summation of an infinite series will often be changed without explicit justification, and the convergence of the infinite series will not be verified. Such justification exists, however, and can be found in Ref. 2.

1. REPRESENTATIONS OF $\mathcal{T}_6$

Just as in I, we study irreducible representations of the 6-dimensional complex Lie algebra $\mathcal{T}_6$. This Lie algebra is defined by the commutation relations

$$\begin{aligned}
&[j^3, j^\pm] = \pm j^\pm, \quad [j^+, j^-] = 2j^3,\\
&[j^3, p^\pm] = [p^3, j^\pm] = \pm p^\pm,\\
&[j^+, p^+] = [j^-, p^-] = [j^3, p^3] = \mathcal{O}, \qquad (1.1)\\
&[j^+, p^-] = [p^+, j^-] = 2p^3,\\
&[p^3, p^\pm] = [p^+, p^-] = \mathcal{O}.
\end{aligned}$$

Here, the elements j^+, j^-, j^3 generate a subalgebra of $\mathcal{T}_6$ isomorphic to $sl(2)$, while p^+, p^-, p^3 generate a 3-dimensional Abelian ideal of $\mathcal{T}_6$.

[1] W. Miller, J. Math. Phys. **9**, 1163 (1968) (preceding paper).

[2] F. W. Schäfke, *Einführung in die Theorie der Speziellen Funktionen der Mathematischen Physik* (Springer-Verlag, Berlin, 1963), Chap. 8.

The 6-parameter Lie group T_6 consists of elements $\{\mathbf{w}, g\}$,

$$\mathbf{w} = (\alpha, \beta, \gamma) \in \mathbb{C}^3,$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \quad ad - bc = 1,$$

with group multiplication

$$\{\mathbf{w}, g\}\{\mathbf{w}', g'\} = \{\mathbf{w} + g\mathbf{w}', gg'\}, \qquad (1.2)$$

where

$$g\mathbf{w} = (a^2\alpha - b^2\beta + ab\gamma, -c^2\alpha + d^2\beta - cd\gamma, 2ac\alpha - 2bd\beta + (bc + ad)\gamma). \quad (1.3)$$

The identity element of T_6 is $\{\mathbf{0}, \mathbf{e}\}$, where $\mathbf{0} = (0, 0, 0)$ and $\mathbf{e}$ is the 2×2 identity matrix. As mentioned in I, $\mathcal{T}_6$ is the Lie algebra of T_6 and a neighborhood of $\mathcal{O}$ in $\mathcal{T}_6$ can be mapped diffeomorphically onto a neighborhood of $\{\mathbf{0}, \mathbf{e}\}$ in T_6 by means of the relation

$$\{\mathbf{w}, g\} = \exp(\alpha p^+ + \beta p^- + \gamma p^3) \exp(-b/d j^+) \times \exp(-cd j^-) \exp(-2 \ln d j^3). \quad (1.4)$$

Here "exp" is the exponential map from $\mathcal{T}_6$ to T_6.

Let V be a complex abstract vector space and ρ a representation of $\mathcal{T}_6$ by linear operators on V. Set

$$\rho(p^\pm) = P^\pm, \quad \rho(p^3) = P^3,$$
$$\rho(j^\pm) = J^\pm, \quad \rho(j^3) = J^3.$$

The linear operators $P^\pm$, P^3, $J^\pm$, J^3 satisfy commutation relations analogous to Eqs. (1.1), where $[A, B] = AB - BA$ for linear operators A and B on V. The operators

$$\mathbf{P} \cdot \mathbf{P} = -P^+P^- - P^3P^3,$$
$$\mathbf{P} \cdot \mathbf{J} = -\tfrac{1}{2}(P^+J^- + P^-J^+) - P^3J^3$$

on V are of special interest, since they have the property

$$[\mathbf{P} \cdot \mathbf{P}, \rho(\alpha)] = [\mathbf{P} \cdot \mathbf{J}, \rho(\alpha)] = 0$$

for all $\alpha \in \mathcal{T}_6$. These two operators turn out to be multiples of the identity operator whenever ρ is one of the irreducible representations of $\mathcal{T}_6$ to be studied in this paper.

Let $\omega \neq 0$ and q be complex numbers. Among the known irreducible representations of $\mathcal{T}_6$,[3,4] we shall examine the following:

(1) $\uparrow_3(\omega, q)$

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u - 1, u - 2, \cdots; u = -q, -q + 1, -q + 2, \cdots$; and $2q$ is not an integer.

[3] W. Miller, *On Lie Algebras and Some Special Functions of Mathematical Physics*, American Mathematical Society Memoir, No. 50 (Providence, 1964).

[4] W. Miller, *Lie Theory and Special Functions* (Academic Press Inc., New York, 1968), Chaps. 5, 6.

(2) $\uparrow_4(\omega, q)$

There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u-1, \cdots, -u+1, -u$; $u = -q, -q+1, \cdots$; and $-2q$ is a nonnegative integer.

(3) $R_3(\omega, q, u_0)$

Here q and u_0 are complex numbers such that $0 \leq \mathrm{Re}\, u_0 < 1$, and none of $u_0 \pm q$ or $2u_0$ is an integer. There is a countable basis $\{f_m^{(u)}\}$ for V such that $m = u, u-1, u-2, \cdots$, and $u = u_0, u_0 \pm 1, u_0 \pm 2, \cdots$.

Corresponding to each of the above representations, the action of the infinitesimal operators on the basis vectors $f_m^{(u)}$ is given by

$$J^3 f_m^{(u)} = m f_m^{(u)}, \quad J^+ f_m^{(u)} = (m-u) f_{m+1}^{(u)}, \quad J^- f_m^{(u)} = -(m+u) f_{m-1}^{(u)}, \tag{1.5}$$

$$P^3 f_m^{(u)} = \frac{\omega(u-q+1)}{(2u+1)(u+1)} f_m^{(u+1)} + \frac{m\omega q}{u(u+1)} f_m^{(u)} - \frac{\omega(u+q)(u+m)(u-m)}{u(2u+1)} f_m^{(u-1)}, \tag{1.6}$$

$$P^+ f_m^{(u)} = \frac{\omega(u-q+1)}{(2u+1)(u+1)} f_{m+1}^{(u+1)} - \frac{(u-m)\omega q}{u(u+1)} f_{m+1}^{(u)} - \frac{\omega(u+q)(u-m)(u-m-1)}{(2u+1)u} f_{m+1}^{(u-1)}, \tag{1.7}$$

$$P^- f_m^{(u)} = -\frac{\omega(u-q+1)}{(2u+1)(u+1)} f_{m-1}^{(u+1)} - \frac{(u+m)\omega q}{u(u+1)} f_{m-1}^{(u)} + \frac{\omega(u+q)(u+m)(u+m-1)}{(2u+1)u} f_{m-1}^{(u-1)}, \tag{1.8}$$

$$\mathbf{P} \cdot \mathbf{P} f_m^{(u)} = -\omega^2 f_m^{(u)} \not\equiv 0, \quad \mathbf{P} \cdot \mathbf{J} f_m^{(u)} = -\omega q f_m^{(u)}. \tag{1.9}$$

[If a vector $f_m^{(u)}$ on the right-hand side of one of the expressions (1.5)–(1.9) does not belong to the representation space, we set this vector equal to zero.]

The reader can verify that the operators defined by expressions (1.5)–(1.8) do satisfy the commutation relations (1.1) and determine the irreducible representations of $\mathcal{E}_6$ listed above. Corresponding to a fixed value of u, the vectors $\{f_m^{(u)}\}$ form a basis for an irreducible representation of the subalgebra $sl(2)$ of $\mathcal{E}_6$. Each such representation of $sl(2)$ induced by $\uparrow_4(\omega, q)$ has dimension $2u+1$ and is denoted by $D(2u)$. Each irreducible representation of $sl(2)$, induced by $\uparrow_3(\omega, q)$ or $R_3(\omega, q, u_0)$, is infinite-dimensional and is denoted by $\downarrow_u$. The notation for the representations in classes (1)–(3) is taken from Ref. 4. A detailed analysis of the representation $D(2u)$ and $\downarrow_u$ is also given in this reference. Note that the representations $\rho_0(\omega)$, $\rho_\mu(\omega)$, studied in I, are identical with the representations $\uparrow_4(\omega, 0)$, $R_3(\omega, 0, \mu)$ presented here.

In analogy with the procedure carried out in I, we will analyze the relationship between the representations in classes (1)–(3) and the special functions of mathematical physics. That is, we will look for models of these abstract representations ρ such that the infinitesimal operators $\rho(\alpha)$, $\alpha \in \mathcal{E}_6$, are linear differential operators acting on a space V of analytic functions in n complex variables. The basis vectors $\{f_m^{(u)}\}$ are then analytic functions and expressions (1.5)–(1.8) are differential recursion relations for these "special" functions. In addition, we will extend each of our Lie-algebra representations of $\mathcal{E}_6$ to a local group representation of T_6. Each such local representation is defined by linear operators $\mathbf{T}(h)$, $h \in T_6$, acting on V and satisfying the group property $\mathbf{T}(h)\mathbf{T}(h') = \mathbf{T}(hh')$ for h, h' in a sufficiently small neighborhood of the identity. We will compute the matrix elements of $\mathbf{T}(h)$ with respect to the basis $\{f_m^{(u)}\}$. The group property then immediately yields addition theorems for these matrix elements. The addition theorems so obtained provide identities relating Bessel functions, Whittaker functions, and Jacobi polynomials.

2. MODELS OF THE REPRESENTATIONS

All possible models of the Lie-algebra representations in classes (1)–(3) are known in which the basis space consists of functions of one or two complex variables.[4] In fact, there is only one such model ($n = 2$):

$$\text{Model A} \quad J^3 = t\frac{\partial}{\partial t}, \quad J^+ = -t\frac{\partial}{\partial z}, \quad J^- = t^{-1}\left((1-z^2)\frac{\partial}{\partial z} - 2zt\frac{\partial}{\partial t} + 2q\right), \tag{2.1}$$
$$P^+ = \omega t, \quad P^- = \omega(1-z^2)t^{-1}, \quad P^3 = \omega z.$$

Here z, t are complex variables, and ω, q are fixed complex constants. It is easy to verify that operators (2.1) satisfy the commutation relations (1.1). Furthermore, we have

$$\mathbf{P} \cdot \mathbf{P} \equiv -\omega^2, \quad \mathbf{P} \cdot \mathbf{J} \equiv -\omega q.$$

Corresponding to this model, the basis vectors $f_m^{(u)}$ are defined up to a multiplicative constant by expressions (1.5)–(1.8), and may be given by

$$f_m^{(u)}(z, t) = \frac{(u-m)!\,\Gamma(u+m+1)}{\Gamma(u-q+1)2^m} P_{u-m}^{(m-q,m+q)}(z) t^m, \tag{2.2}$$

where $\Gamma(x)$ is the gamma function and $P_n^{(\alpha,\beta)}$ is a

Jacobi polynomial.[4] The possible values of u, m, q, ω depend on the representation in classes (1)–(3) with which we are concerned, and these values have been listed in Sec. 1.

By substituting the Model A operators and basis vectors into expressions (1.5)–(1.8), we obtain the following well-known recursion relations obeyed by Jacobi polynomials:

$$\frac{d}{dz}P_n^{(\alpha,\beta)}(z) = \tfrac{1}{2}(\alpha+\beta+n+1)P_{n-1}^{(\alpha+1,\beta+1)}(z),$$

$$\left[(1-z^2)\frac{d}{dz} - (\alpha+\beta)z + \beta - \alpha\right]P_n^{(\alpha,\beta)}(z) = -2(n+1)P_{n+1}^{(\alpha-1,\beta-1)}(z), \quad (1.5')$$

$$zP_n^{(\alpha,\beta)}(z) = \frac{2(n+1)(\alpha+\beta+n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}P_{n+1}^{(\alpha,\beta)}(z) + \frac{(\beta^2-\alpha^2)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}P_n^{(\alpha,\beta)}(z) + \frac{2(n+\alpha)(n+\beta)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}P_{n-1}^{(\alpha,\beta)}(z), \quad (1.6')$$

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+\beta+n+1)(\alpha+\beta+n+2)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}P_n^{(\alpha+1,\beta+1)}(z) + \frac{(\alpha-\beta)(\alpha+\beta+n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}P_{n-1}^{(\alpha+1,\beta+1)}(z) - \frac{(\alpha+n)(\beta+n)}{(\alpha+\beta+2n)(\alpha+\beta+2n+1)}P_{n-2}^{(\alpha+1,\beta+1)}(z), \quad (1.7')$$

$$\tfrac{1}{4}(1-z^2)P_n^{(\alpha,\beta)}(z) = -\frac{(n+2)(n+1)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}P_{n+2}^{(\alpha-1,\beta-1)}(z) + \frac{(\alpha-\beta)(n+1)}{(\alpha+\beta+2n)(\alpha+\beta+2n+2)}P_{n+1}^{(\alpha-1,\beta-1)}(z) + \frac{(\alpha+n)(\beta+n)}{(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)}P_n^{(\alpha-1,\beta-1)}(z), \quad (1.8')$$

valid for $n = 0, 1, 2, \cdots$, and $\alpha, \beta \in \not{C}$.

Those representations of $\mathcal{G}_6$, for which $q = 0$, have a model (Model B) in terms of differential operators in three complex variables. Model B was constructed and studied in I. If $q \neq 0$, there is no model in three complex variables. However, in Sec. 8 we will construct a model (Model C) in terms of differential operators acting on spinor-valued functions in three complex variables. The special functions obtained from Model C are closely related to the spinor-valued solutions of the wave equation in 3-space.

3. ANALYSIS OF THE MODELS

The following section contains several auxilliary lemmas which will enable us to extend the representations $\uparrow_3(\omega, q)$, $\uparrow_4(\omega, q)$, and $R_3(\omega, q, u_0)$ of $\mathcal{G}_6$ to local group representations of T_6. Throughout this section it is assumed that the operators $J^\pm, J^3, P^\pm, P^3$ and the basis vectors $f_m^{(u)}$ correspond to one of the irreducible Lie-algebra representations listed above. The results will be formally the same for all of these representations, the only difference being the allowable values of u, m, q, and ω.

Lemma 1: Let I be the identity operator on V:

$$(\omega I - P^3)^k f_u^{(u)} = \frac{(2\omega)^k k!\,\Gamma(u-q+k+1)\Gamma(2u+1)}{\Gamma(u-q+1)} \times \sum_{n=0}^{k} \frac{(-1)^n(2u+2n+1)}{n!\,(k-n)!\,\Gamma(2u+n+k+2)} f_u^{(u+n)}.$$

Proof: Use of expression (1.6) and induction on k.

Corollary 1: Let $\alpha, \beta \in \not{C}$ and k a nonnegative integer:

$$\left(\frac{1-z}{2}\right)^k = k!\,\Gamma(k+\alpha+1) \times \sum_{n=0}^{k} \frac{\Gamma(\alpha+\beta+n+1)(\alpha+\beta+2n+1)(-1)^n}{(k-n)!\,\Gamma(n+\alpha+1)\Gamma(\alpha+\beta+n+k+2)} \times P_n^{(\alpha,\beta)}(z).$$

Proof: This is the content of Lemma 1 when it is applied to Model A.

As is well known,[5] the Jacobi polynomials are related to the Gauss hypergeometric functions by the formula

$$P_n^{(\gamma,\delta)}(z) = \binom{n+\gamma}{n}\, {}_2F_1\left(-n, \gamma+\delta+n+1; \gamma+1; \frac{1-z}{2}\right). \quad (3.1)$$

From this expression and Corollary 1 it is a straightforward computation to obtain the identity

$$P_n^{(\gamma,\delta)}(z) = \sum_{k=0}^{n} \frac{\Gamma(\gamma+\delta+n+k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\gamma+n+1)}{\Gamma(\alpha+\beta+2k+1)\Gamma(\gamma+\delta+n+1)\Gamma(\gamma+k+1)(n-k)!} \times {}_3F_2(k-n, \gamma+\delta+n+k+1, \alpha+k+1; \gamma+k+1, \alpha+\beta+2k+2; 1)P_k^{(\alpha,\beta)}(z), \quad (3.2)$$

expressing an arbitrary Jacobi polynomial $P_n^{(\gamma,\delta)}(z)$ as a linear combination of the polynomials $P_k^{(\alpha,\beta)}(z)$.

[5] W. Magnus, F. Oberhettinger, and R. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics* (Springer-Verlag, New York, 1966), 3rd ed.

Passing from Model A back to our abstract representation, we obtain:

Lemma 2: Let $\gamma, \delta \in \mathbb{C}$ and n a nonnegative integer.

$$P_n^{(\gamma,\delta)}(\omega^{-1}P^3)f_u^{(u)} = \sum_{k=0}^{n} \frac{\Gamma(2u+1)\Gamma(u-q+k+1)}{k!\,(n-k)!\,\Gamma(u-q+1)} \times \frac{\Gamma(\gamma+\delta+n+k+1)\Gamma(\gamma+n+1)}{\Gamma(2u+2k+1)\Gamma(\gamma+\delta+n+1)\Gamma(\gamma+k+1)} \times {}_3F_2(k-n, \gamma+\delta+n+k+1, u-q+k+1; \gamma+k+1, 2u+2k+2; 1) f_u^{(u+k)}.$$

Although the function ${}_3F_2(1)$ appears complicated, it can be explicitly evaluated in several interesting special cases. If $\alpha = \gamma$ in Eq. (3.2), then ${}_3F_2(1)$ reduces to the form ${}_2F_1(1)$. Using the well-known formula[5]

$$ {}_2F_1(a, -n; c; 1) = \frac{\Gamma(c-a+n)\Gamma(c)}{\Gamma(c-a)\Gamma(c+n)}, \quad n = 0, 1, 2, \cdots, $$

we find

$$P_n^{(\alpha,\delta)}(z) = \sum_{k=0}^{n} \frac{\Gamma(\alpha+\delta+n+k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\alpha+n+1)}{\Gamma(\alpha+\delta+n+1)\Gamma(\alpha+k+1)\Gamma(\alpha+\beta+k+n+2)} \times \frac{(\alpha+\beta+2k+1)}{(n-k)!} \frac{\Gamma(\beta-\delta+1)}{\Gamma(\beta-\delta+k-n+1)} \times P_n^{(\alpha,\beta)}(z). \tag{3.3}$$

If $\beta = \delta$, then ${}_3F_2$ is Saalschutzian[6] and ${}_3F_2(1)$ can be explicitly evaluated to yield

$$P_n^{(\gamma,\beta)}(z) = \sum_{k=0}^{n} \frac{\Gamma(\gamma+\beta+n+k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\beta+n+1)}{\Gamma(\gamma+\beta+n+1)\Gamma(\alpha+\beta+k+n+2)} \times \frac{(\alpha+\beta+2k+1)(-1)^{n-k}\Gamma(\alpha-\gamma+1)}{\Gamma(\beta+k+1)(n-k)!\,\Gamma(\alpha-\gamma+k-n+1)} \times P_k^{(\alpha,\beta)}(z). \tag{3.4}$$

Finally, if $\alpha = \beta$ and $\gamma = \delta$, we can use Watson's theorem[6]

$$ {}_3F_2\left(a, b, c; \frac{1}{2} + \frac{a}{2} + \frac{b}{2}, 2c; 1\right) = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma\left(\frac{a+b+1}{2}\right)\Gamma\left(\frac{1-a-b+2c}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)\Gamma\left(\frac{1-a+2c}{2}\right)\Gamma\left(\frac{1-b+2c}{2}\right)}, $$

with the result

$$P_n^{(\gamma,\gamma)}(z) = \sum_{k=0}^{n} \frac{\Gamma(2\gamma+n+k+1)\Gamma(\gamma+\alpha+k+1)\Gamma(\gamma+n+1)\Gamma(\frac{1}{2})}{\Gamma(2\alpha+2k+1)\Gamma(2\gamma+n+1)\Gamma(\gamma+k+1)(n-k)!} \times \frac{\Gamma(\alpha+k+\frac{1}{2})\Gamma(\gamma+k+1)\Gamma(\alpha-\gamma+1)P_k^{(\alpha,\alpha)}(z)}{\Gamma\left(\frac{k-n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{2\gamma+n+k+2}{2}\right)\Gamma\left(\frac{2\alpha+n+k+3}{2}\right)\Gamma\left(\frac{2\alpha-2\gamma+k-n+2}{2}\right)}. \tag{3.5}$$

Since $\Gamma[(k-n)/2+\frac{1}{2}]$ occurs in the denominator of the right-hand side of Eq. (3.5), the coefficient of $P_k^{(\alpha,\alpha)}(z)$ is nonzero only if $n-k$ is an even integer. Because of the well-known identity[5]

$$C_n^{\lambda}(z) = \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(2\lambda+n)}{\Gamma(2\lambda)\Gamma(\lambda+n+\frac{1}{2})} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(z),$$

expression (1.16) is readily seen to be equivalent to Corollary 3 of I.

[6] L. J. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, Cambridge, England, 1966), Chap. 2.

In the following sections we shall find it useful to expand the product $P_n^{(\alpha,\beta)}(z)P_l^{(\alpha,\beta)}(z)$ as a linear combination of Jacobi polynomials $P_k^{(\alpha,\beta)}(z)$:

$$P_n^{(\alpha,\beta)}(z)P_l^{(\alpha,\beta)}(z) = \sum_{k=0}^{n+l} E^{\alpha,\beta}(n, l; k)P_k^{(\alpha,\beta)}(z). \tag{3.6}$$

The coefficient $E^{\alpha,\beta}(\cdot)$ can be obtained by first using Eq. (3.1) to express the left-hand side of Eq. (3.6) as a polynomial in $(1-z)$ and then using Corollary 1 to write the resulting polynomial as a linear combination of $P_k^{(\alpha,\beta)}(z)$. The result is

$$E^{\alpha,\beta}(n,l;k)=\sum_{r=0}^{n}\sum_{s=0}^{l}\frac{\Gamma(\alpha+n+1)\Gamma(\alpha+l+1)\Gamma(\alpha+\beta+k+1)}{\Gamma(\alpha+\beta+n+1)\Gamma(\alpha+\beta+l+1)}$$
$$\times\frac{\Gamma(\alpha+\beta+n+r+1)\Gamma(\alpha+\beta+l+s+1)\Gamma(\alpha+r+s+1)}{\Gamma(\alpha+k+1)\Gamma(\alpha+r+1)\Gamma(\alpha+s+1)\Gamma(\alpha+\beta+r+s+k+2)}\times\frac{(-1)^k(\alpha+\beta+2k+1)(r+s)!}{r!(n-r)!s!(l-s)!(r+s-k)!}. \tag{3.7}$$

This is not a very enlightening expression. However, in certain special cases, the coefficients can be evaluated very simply. For example, as was shown in I, if $\alpha=\beta=\lambda-\frac{1}{2}$, then Eq. (3.6) becomes

$$C_n^\lambda(z)C_l^\lambda(z)=\sum_{k=0}^{\min(n,l)}\frac{(n+l-2k)!(\lambda+n+l-2k)}{k!(n-k)!(l-k)!}$$
$$\times\frac{\Gamma(2\lambda+n+l-k)\Gamma(\lambda+l-k)\Gamma(\lambda+k)\Gamma(\lambda+n-k)}{\Gamma(2\lambda+n+l-2k)\Gamma(\lambda+n+l-k+1)\Gamma^2(\lambda)}\times C^\lambda_{n+l-2k}(z).$$

The reader can undoubtedly derive other formulas for $E^{\alpha,\beta}(\cdot)$, some of which are more transparent than Eq. (3.7). In particular, it is not difficult to show [by means of the recursion relation (1.6)] that $E^{\alpha,\beta}(n,l;k)=0$, unless $n+l\geq k\geq|n-l|$. Here we will merely point out the connection between these coefficients and the representation theory of $\mathcal{G}_6$.

Expression (3.6) has been established by direct computation for Model A, but it implies the existence of a similar expression obeyed by the abstract representations of $\mathcal{G}_6$ and by any model of these representations.

Lemma 3:

$$P_n^{(m-q,m+q)}(\omega^{-1}P^3)f_m^{(m+l)}=\sum_{k=0}^{n+l}\frac{l!\,\Gamma(2m+l+1)\Gamma(m-q+k+1)}{k!\,\Gamma(2m+k+1)\Gamma(m-q+l+1)}\times E^{m-q,m+q}(n,l;k)f_m^{(m+k)},\quad n,l=0,1,2,\cdots.$$

Corollary 2:

$$P_n^{(m-q,m+q)}(\omega^{-1}P^3)f_m^{(m)}=\frac{\Gamma(2m+1)\Gamma(m-q+n+1)}{n!\,\Gamma(2m+n+1)\Gamma(m-q+1)}f_m^{(m+n)}.$$

Lemma 4:

$$(\tfrac{1}{2}P^+)^l f_m^{(m+n)}=\omega^l\sum_{k=\max(n-2l,0)}^{n}\frac{n!\,\Gamma(m-q+l+k+1)\Gamma(2m+n+k+1)}{k!\,(n-k)!\,\Gamma(m-q+k+1)\Gamma(2m+2l+2k+1)}$$
$$\times{}_3F_2(k-n,2m+n+k+1,m-q+l+k+1;m-q+k+1,2m+2l+2k+2;1)f_{m+l}^{(m+l+k)}.$$

Proof: It follows from Eq. (3.2) that the lemma is true for Model A. Hence, it must be true for any model.

Let P be a linear operator on V and $a\in\mathcal{C}$. Define $\exp(aP)$ as the formal sum $\sum_{k=0}^\infty(a^k/k!)(P)^k$. We will use our lemmas to compute the operators $\exp(aP^3)$, $\exp(aP^+)$, and $\exp(aP^-)$ on V. These results are purely formal when applied to the abstract representations of $\mathcal{G}_6$. However, when applied to models of these representations, they have a rigorous justification.

Lemma 5:

$$\exp(a(P^3-\omega I))f_u^{(u)}=\sum_{n=0}^\infty\frac{\Gamma(2u+1)\Gamma(u+n-q+1)}{n!\,\Gamma(u-q+1)\Gamma(2u+2n+1)}(2a\omega)^n\times{}_1F_1(m-q+n+1;2m+2n+2;-2a\omega)f_u^{(u+n)}.$$

Proof: This is a direct consequence of Lemma 1.

It will be shown later that Lemma 5 is valid for Model A. Thus, we have:

Corollary 3: Let $\alpha,\beta,a\in\mathcal{C}$. Then

$$e^{az}=(2a)^{-1-(\alpha+\beta)/2}\times\sum_{n=0}^\infty\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}M_{\chi,\mu}(2a)P_n^{(\alpha,\beta)}(z),$$

where $\chi=(\alpha-\beta)/2$, $\mu=n+(\alpha+\beta+1)/2$, and

$$M_{\chi,\mu}(a)=e^{a/2}a^{\mu+\frac{1}{2}}{}_1F_1(\mu+\chi+\tfrac{1}{2};1+2\mu;-a)$$

is a Whittaker function.[5]

Corollary 4:

$$\exp(aP^3)f_u^{(u)} = (2a\omega)^{-1-u}\sum_{n=0}^{\infty}\frac{\Gamma(2u+1)\Gamma(u-q+n+1)}{n!\,\Gamma(u-q+1)\Gamma(2u+2n+1)} \times M_{-q,u+n+\frac{1}{2}}(2a\omega)f_u^{(u+n)}.$$

Corollary 5:

$$\exp(a\omega P^3) = (2a\omega)^{-1-(\alpha+\beta)/2}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+\beta+n+1)}{\Gamma(\alpha+\beta+2n+1)}M_{\chi,\mu}(2a\omega) \times P_n^{(\alpha,\beta)}(P^3),$$

$$\chi = (\alpha-\beta)/2,\quad \mu = n + (\alpha+\beta+1)/2.$$

4. LOCAL REPRESENTATIONS OF T_6

Since the Model A operators (1.10) satisfy the commutation relations of $\mathcal{C}_6$, they induce a local representation of T_6 by operators $\mathbf{T}(h)$, $h \in T_6$, acting on the space of analytic functions in two complex variables. The details necessary for the computation of $\mathbf{T}(h)$ have been listed elsewhere.[4,7] We present only the results. According to the group multiplication law, it follows that

$$\mathbf{T}(h) = \mathbf{T}(\mathbf{w}; g) = \mathbf{T}(\mathbf{w}; \mathbf{e})\mathbf{T}(\mathbf{0}; g),$$

where

$$h = \{\mathbf{w}, g\},\quad \mathbf{w} = (\alpha, \beta, \gamma) \in \mathcal{C}^3,$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2).$$

Let f be an analytic function defined in a neighborhood of some point $(z, t) \in \mathcal{C}^2$ $(t \neq 0)$. Then

$$[\mathbf{T}(\mathbf{w}; \mathbf{e})f](z,t) = [\exp(\alpha P^+ + \beta P^- + \gamma P^3)f](z,t) = \exp \omega[\alpha t + \beta(1-z^2)t^{-1} + \gamma z]f(z,t), \quad (4.1)$$

$$[\mathbf{T}(\mathbf{0}; g)f](z,t) = [\exp(-b/dJ^+)\exp(-cdJ^-) \times \exp(-2\ln dJ^3)f](z,t)$$
$$= \left(\frac{at + c(z-1)}{at + c(z+1)}\right)^q f\left(z(1+2bc) + abt + \frac{cd}{t}(z^2-1),\ a^2t + 2acz + \frac{c^2}{t}(z^2-1)\right). \quad (4.2)$$

These operators satisfy the group property

$$\mathbf{T}(hh')f = \mathbf{T}(h)[\mathbf{T}(h')f], \quad (4.3)$$

whenever both sides of this expression are well defined.

[7] H. W. Guggenheimer, *Differential Geometry* (McGraw-Hill Book Co., Inc., New York, 1963), Chap. 7.

5. MATRIX ELEMENTS FOR $\uparrow_4(\omega, q)$

We are now able to compute the matrix elements of the group representations of T_6 induced by the representations $\uparrow_4(\omega, q)$. The restrictions of these representations to the real Euclidean group in 3-space are known to be unitary and irreducible, and have been studied in detail elsewhere.[4,8,9]

Throughout this section, $u, v = -q, -q+1, \cdots$; and $-2q$ is a nonnegative integer. Furthermore, m and n will range over the values $m = -u, -u+1, \cdots, u-1, u$; and $n = -v, -v+1, \cdots, v-1, v$. The matrix elements $\{v, n|\,\mathbf{w}, g\,|u, m\}$ of $\uparrow_4(\omega, q)$ are defined by

$$\mathbf{T}(\mathbf{w}, g)f_m^{(u)} = \sum_{2v=-2q}^{\infty}\sum_{n=-v}^{v}\{v, n|\,\mathbf{w}, g\,|u, m\}f_n^{(v)}, \quad (5.1)$$

where the operator $\mathbf{T}(\mathbf{w}, g)$ and basis functions $f_m^{(u)}$ refer to Model A. According to Ref. 2, the Jacobi polynomials (2.2) form an analytic basis for the representation space. That is, the functions $\mathbf{T}(\mathbf{w}, g)f_m^{(u)}$ can be expressed uniquely as a linear combination of basis functions $f_n^{(v)}$ uniformly convergent in a suitable domain. The coefficients in the expansion are bounded linear functionals of the argument $\mathbf{T}(\mathbf{w}, g)f_m^{(u)}$ in the topology of uniform convergence on compact sets.

Under these conditions, the matrix elements (5.1) are model-independent: They are uniquely determined by relations (1.5)–(1.8) and are the same for every model of $\uparrow_4(\omega, q)$ which has an analytic basis.[4] We can compute the matrix elements using either (1.5)–(1.8) or Model A and our results will automatically be valid for any other model of $\uparrow_4(\omega, q)$. Moreover, the relation

$$\mathbf{T}(\mathbf{w}, g)\mathbf{T}(\mathbf{w}', g') = \mathbf{T}(\mathbf{w} + g\mathbf{w}', gg')$$

implies the addition theorem[4]

$$\sum_{2v'=-2q}^{\infty}\sum_{n'=-v'}^{v'}\{v, n|\,\mathbf{w}, g\,|v', n'\}\{v', n'|\,\mathbf{w}', g'\,|u, m\} = \{v, n|\,\mathbf{w} + g\mathbf{w}', gg'\,|u, m\}. \quad (5.2)$$

The matrix elements $\{v, n|\,\mathbf{0}, g\,|u, m\}$ are uniquely determined by the J operators (1.5) and depend entirely on the representation theory of $Sl(2)$. Indeed, for fixed u the vectors $f_m^{(u)}$ form a basis for the $(2u+1)$-dimensional irreducible representation of $Sl(2)$. The matrix elements of these finite-dimensional

[8] N. Y. Vilenkin, E. L. Akim, and A. A. Levin, Dokl. Akad. Nauk SSSR **112**, 987 (1957).

[9] W. Miller, Commun. Pure Appl. Math. **17**, 527 (1964).

representations are well known[4]:

$$\{v, n|\,\mathbf{0}, g\,|u, m\} = \frac{d^{u-n}a^{u+m}b^{n-m}(u-m)!}{(u-n)!} \times {}_2F_1\frac{(n-u, -m-u; n-m+1; bc/ad)}{\Gamma(n-m+1)}\delta_{v,u}$$
$$= d^{u-m}a^{u+n}c^{m-n}\frac{(u+m)!}{(u+n)!} \times \frac{{}_2F_1(m-u, -n-u; m-n+1; bc/ad)}{\Gamma(m-n+1)}\delta_{v,u},$$
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \quad ad - bc = 1. \quad (5.3)$$

Because of the relation

$$\lim_{c\to -k}\frac{{}_2F_1(a, b; c; z)}{\Gamma(c)} = \frac{a(a+1)\cdots(a+k)b(b+1)\cdots(b+k)}{(k+1)!} \times z^{k+1}{}_2F_1(a+k+1, b+k+1; k+2; z), \quad k = 0, 1, 2, \cdots,$$

expressions (5.3) make sense for all permissible values of m and n. Note that the hypergeometric functions can be expressed in terms of Jacobi polynomials.

In terms of Model A, the identity

$$\mathbf{T}(\mathbf{0}; g)f_m^{(u)} = \sum_{n=-u}^{u}\{u, n|\,\mathbf{0}, g\,|u, m\}f_n^{(u)} \quad (5.4)$$

implies

$$[at - c(z-1)]^{m+q}[at + c(z+1)]^{m-q}P_{u-m}^{(m-q,m+q)} \times \left[z(1+2bc) + abt + \frac{cd}{t}(z^2-1)\right]$$
$$= \sum_{n=-u}^{u} d^{u-m}a^{u+n}(2c)^{m-n}\frac{(u-n)!}{(u-m)!} \times \frac{{}_2F_1(m-u, -n-u; m-n+1; bc/ad)}{\Gamma(m-n+1)} \times P_{u-n}^{(n-q,n+q)}(z)t^{n+m},$$
$$\left|\frac{c(z\pm 1)}{at}\right| < 1, \quad ad - bc = 1. \quad (5.5)$$

When $u = m$, Eq. (5.5) simplifies to

$$[1 - c(z-1)]^{u+q}[1 + c(z+1)]^{u-q} = \sum_{n=-u}^{u}(2c)^{u-n}P_{u-n}^{(n-q,n+q)}(z), \quad |c(z\pm 1)| < 1.$$

Since the Model A functions $f_m^{(u)}(z, t)$ form an analytic basis, Lemma 5 and its corollaries are rigorously true for Model A. Thus,

$$\mathbf{T}(0, 0, \gamma; \mathbf{e})f_m^{(u)} = \exp(\gamma P^3)f_m^{(u)}$$
$$= (2\gamma)^{-1-m}\sum_{k=0}^{\infty}\frac{\Gamma(2m+k+1)}{\Gamma(2m+2k+1)}M_{-q,m+k+\frac{1}{2}}(2\gamma) \times P_k^{(m-q,m+q)}(\omega^{-1}P^3)f_m^{(u)}$$
$$= (2\gamma)^{-1-m}\sum_{j=-\infty}^{\infty}f_m^{(u+j)}\sum_{k=0}^{\infty}\frac{\Gamma(2m+k+1)}{\Gamma(2m+2k+1)} \times M_{-q,m+k+\frac{1}{2}}(2\gamma)\frac{(u-m)!\,(u+m)!\,(u-q+j)!}{(u-m+j)!\,(u+m+j)!\,(u-q)!} \times E^{m-q,m+q}(k, u-m; u-m+j)$$

and

$$\{v, n|\,0, 0, \gamma; \mathbf{e}\,|u, m\} = \delta_{n,m}(2\gamma)^{-1-m} \times \sum_k\frac{(2m+k)!\,(u-m)!\,(u+m)!\,(v-q)!}{(2m+2k)!\,(v-m)!\,(v+m)!\,(u-q)!} \times E^{m-q,m+q}(k, u-m; v-m)M_{-q,m+k+\frac{1}{2}}(2\gamma), \quad (5.6)$$

where the sum is taken over the finite number of values of k such that the summand is defined. In the special case $m = u$ we obtain

$$\{v, n|\,0, 0, \gamma; \mathbf{e}\,|u, u\} = \delta_{n,u}(2\gamma)^{-1-u}\frac{(2u)!\,(v-q)!}{(2v)!\,(v-u)!\,(u-q)!}M_{-q,v+\frac{1}{2}}(2\gamma), \quad \text{if } v \ge u,$$
$$= 0, \quad \text{if } v < u. \quad (5.7)$$

To compute the general matrix element

$$\{v, n|\,\alpha, \beta, \gamma; \mathbf{e}\,|u, m\},$$

we make use of the identity

$$\exp\omega[\alpha t + \beta(1-z^2)t^{-1} + \gamma z] = \sum_{j=0}^{\infty}\sum_{k=-j}^{j}(\pi\omega\rho/2)^{-\frac{1}{2}}(4/\rho)^{|k|}(\alpha)^{(|k|+k)/2}(-\beta)^{(|k|-k)/2} \times \frac{\Gamma(|k|+\frac{1}{2})\Gamma(k+\frac{1}{2})(j-k)!\,(j+\frac{1}{2})}{(j+|k|)!} \times I_{j+\frac{1}{2}}(\omega\rho)C_{j-k}^{|k|+\frac{1}{2}}(\gamma/\rho)C_{j-k}^{k+\frac{1}{2}}(z)(2t)^k, \quad (5.8)$$

which was derived in I. Here $\rho^2 = \gamma^2 + 4\alpha\beta$, $C_j^\lambda(z)$ is a Gegenbauer polynomial and

$$I_\lambda(z) = \frac{(z/2)^\lambda}{\Gamma(1+\lambda)}{}_0F_1(\lambda+1; z^2/4) = \frac{(2z)^{-\frac{1}{2}}2^{-2\lambda}}{\Gamma(1+\lambda)}M_{0,\lambda}(2z)$$

is a modified Bessel function. The right-hand side of Eq. (5.8) is an entire function of αt, β/t, γ, and z. Furthermore, it is a function of ρ^2.

The second identity we will need is related to the representation theory of $SL(2)$:

$$P_{u-m'}^{m'-m,m'+m}(z)P_{v-n'}^{n'-n,n'+n}(z) = \sum_{s=0}^{2\min(u,v)}D(u, m, m'; v, n, n'; s) \times C(u, m; v, n\,|\,u+v-s, m+n) \times C(u, m'; v, n'\,|\,u+v-s, m'+n') \times P_{u+v-s-m'-n'}^{m'+n'-m-n,m'+n'+m+n}(z). \quad (5.9)$$

Here,

$$D(u, m, m'; v, n, n'; s) = \left[\frac{(u-m)!\,(u+m)!\,(v-n)!\,(v+n)!}{(u-m')!\,(u+m')!\,(v-n')!\,(v+n')!} \times \frac{(u+v-s-m'-n')!\,(u+v-s+m'+n')!}{(u+v-s-m-n)!\,(u+v-s+m+n)!}\right]$$

and the $C(\cdot\,;\cdot\mid\cdot)$ are Clebsch–Gordan coefficients. (For a group-theoretic proof of this result see Refs. 4, 10, 11.)

Now, making use of Model A, we have

$$\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e}) f_m^{(u)}(z, t) = \sum_{v,n} \{v, n \mid \alpha, \beta, \gamma; \mathbf{e} \mid u, m\} f_n^{(v)}(z, t) = \exp\left[\omega(\alpha t + \beta(1-z^2)/t + \gamma z)\right] \times \frac{(u-m)!\,(u+m)!}{(u-q)!\,2^m} P_{u-m}^{(m-q,m+q)}(z) t^m. \quad (5.10)$$

Applying the two identities to the right-hand side of this expression, we obtain

$$\begin{aligned}\{v, n \mid \alpha, \beta, \gamma; \mathbf{e} \mid u, m\} &= (\pi\omega\rho/2)^{-\frac{1}{2}}(4/\rho)^{|n-m|} \\ &\times (4\alpha)^{(|n-m|+n-m)/2}\left(\frac{-\beta}{4}\right)^{(|n-m|+m-n)/2} \\ &\times \frac{\Gamma(|n-m|+\frac{1}{2})\Gamma(n-m+\frac{1}{2})(n-m)!}{(2n-2m)!} \\ &\times \sum_s G(u, m; v, n; q, s) C(v-u+s; u, q \mid v, q) \\ &\times C(v-u+s, n-m; u, m \mid v, n) \\ &\times I_{v-u+s+\frac{1}{2}}(\omega\rho) C_{v-u+s-|n-m|}^{|n-m|+\frac{1}{2}}(\gamma/\rho), \end{aligned} \quad (5.11)$$

where

$$G(u, m; v, n; q, s) = \frac{(v-u+s)}{(v-u+s+|n-m|)!} \times \left[\frac{(u+q)!\,(v-u+s-n+m)!\,(v-u+s+n-m)!\,(u-m)!\,(u+m)!\,(v-q)!}{(u-q)!\,(v-k-m)!\,(v+k+m)!\,(v+q)!}\right]^{\frac{1}{2}}.$$

The sum is taken over the finite set of nonnegative integral values for which the summand is defined. These matrix elements are entire functions of α, β, and γ.

By substituting expressions (5.3) and (5.11) for the matrix elements of $\uparrow_4(\omega, q)$ into the addition theorem (5.2), the reader can derive a number of identities relating spherical Bessel functions and Gegenbauer polynomials.

6. MORE MATRIX ELEMENTS

The expressions for matrix elements of $\uparrow_4(\omega, q)$ were rather complicated, and the expressions for matrix elements of $\uparrow_3(\omega, q)$ and $R_3(\omega, q, u_0)$ are even more complicated. Nonetheless, these representations are closely related to a number of important identities in special function theory. In order to keep the computations as simple as possible, we compute directly only a few interesting special cases of the matrix elements of $\uparrow_3(\omega, q)$ and $R_3(\omega, q, u_0)$. (In Sec. 7, however, we obtain expressions for the general matrix elements by relating them to matrix elements of other representations of $\mathcal{C}_6$.)

The matrix elements $\{v, n \mid \mathbf{w}, g \mid u, m\}$ of $\uparrow_3(\omega, q)$ and $R_3(\omega, q, u_0)$ are defined by

$$\mathbf{T}(\mathbf{w}; g) f_m^{(u)}(z, t) = \sum_v \sum_n \{v, n \mid \mathbf{w}; g \mid u, m\} f_n^{(v)}(z, t), \quad (6.1)$$

where the operators and basis functions refer to Model A. [Corresponding to $\uparrow_3(\omega, q)$, the variables assume values u, $v = -q, -q+1, -q+2, \cdots$; $m = u, u-1, u-2, \cdots, n = v, v-1, v-2, \cdots$, where $2q \in \not{C}$ is not an integer. Corresponding to $R_3(\omega, q, u_0)$, $u, v = u_0,\ u_0 \pm 1,\ u_0 \pm 2, \cdots$; $m = u, u-1, u-2, \cdots; n = v, v-1, v-2, \cdots$, where q, u_0 are complex numbers such that $0 \le \operatorname{Re} u_0 < 1$, and none of u_0, $\pm q$, or $2u_0$ is an integer. The formal expressions giving the matrix elements are identical for both classes of representations; the difference between them is merely the different range of values assumed by the variables $u, v, m, n, q, u_0, \omega$.]

It is well known[2] that the Model A functions $f_m^{(u)}(z, t)$ form an analytic basis for the representation space. Hence, the matrix elements are well defined and uniquely determined by the Lie-algebra relations (1.5)–(1.8). Moreover, Lemma 5 and its corollaries are valid.

The action of the operators $J^\pm$, J^3 on the basis vectors $\{f_m^{(u)}\}$ for fixed u, $m = u, u-1, u-2, \cdots$ defines an irreducible representation $\downarrow_u$ on $sl(2)$. This infinite-dimensional representation was studied in Ref. 4, Chap. 5. Its matrix elements are

$$\{v, n \mid \mathbf{0}, g \mid u, m\} = d^{u-n} a^{u+m} b^{n-m} \frac{(u-m)!}{(u-n)!} \times \frac{{}_2F_1(n-u, -m-u; n-m+1; bc/ad)}{\Gamma(n-m+1)} \delta_{v,u}$$

[10] G. Y. Lyubarskii, *The Application of Group Theory to Physics* (Pergamon Press, Oxford, 1960), English Transl., Chap. 10.

[11] N. Y. Vilenkin, *Special Functions and Theory of Group Representations* (Izd. Nauka., Moscow, 1965).

$$= d^{u-m}a^{u+n}c^{m-n}\frac{\Gamma(u+m+1)}{\Gamma(u+n+1)} \times \frac{{}_2F_1(m-u,-n-u;m-n+1;bc/ad)}{\Gamma(m-n+1)}\delta_{v,u},$$

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2), \quad ad - bc = 1. \tag{6.2}$$

The matrix elements define a local representation of the group $SL(2)$. That is, they are defined and satisfy the group representation property only in suitably small neighborhoods of $\mathbf{e}$. These neighborhoods have been determined elsewhere[4] and are usually evident by inspection.

Substituting expression (6.2) into the identity

$$\mathbf{T}(\mathbf{0}, g)f_m^{(u)}(z,t) = \sum_{k=0}^{\infty}\{u, u-k|\,\mathbf{0}, g\,|u, m\}f_{u-k}^{(u)}(z,t),$$

we find

$$t^{m-u}\left(a - \frac{c(z-1)}{t}\right)^{m+q}\left(a + \frac{c(z+1)}{t}\right)^{m-q} \times P_{u-m}^{(m-q,m+q)}\left[z(1+2bc) + abt + \frac{cd}{t}(z^2-1)\right]$$

$$= \sum_{k=0}^{\infty} d^{u-m}a^{2u-k}(2c)^{m-u+k}\frac{k!}{(u-m)!} \times \frac{{}_2F_1(m-u,-2u+k;m-u+k+1;bc/ad)}{\Gamma(m-u+k+1)} \times P_k^{(u-k-q,u-k+q)}(z)t^{-k},$$

$$\left|\frac{c(z\pm 1)}{at}\right| < 1, \quad ad - bc = 1. \tag{6.3}$$

When $u = m$, this relation becomes

$$[1 - c(z-1)]^{u+q}[1 + c(z-1)]^{u-q} = \sum_{k=0}^{\infty}(2c)^kP_k^{(u-k-q,u-k+q)}(z), \quad |c(z\pm 1)| < 1.$$

Matrix elements of the form $\{v, n|\,0, 0, \gamma;\mathbf{e}\,|u, m\}$ can be computed directly from Corollary 5 and Lemma 3. The result is

$$\{v, n|\,0,0,\gamma;\mathbf{e}\,|u,m\} = \delta_{n,m}M_{m;q}^{v,u}(2\gamma\omega)$$
$$= (2\gamma\omega)^{-1-m}\frac{\Gamma(u+m+1)(u-m)!\,\Gamma(v-q+1)}{\Gamma(v+m+1)(v-m)!\,\Gamma(u-q+1)} \times \sum_{n=0}^{\infty}\frac{\Gamma(2m+n+1)}{\Gamma(2m+2n+1)} \times E^{m-q,m+q}(n,u-m,v-m)M_{-q,m+n+\frac{1}{2}}(2\gamma\omega). \tag{6.4}$$

(The sum actually contains only a finite number of nonzero terms.)

The functions $M_{m;q}^{v,u}(\gamma)$, defined by Eq. (6.4), form a generalization of the Whittaker functions $M_{\chi,\mu}(\gamma)$, since

$$M_{u;q}^{v,u}(\gamma) = \frac{\gamma^{-1-u}}{(v-u)!}\frac{\Gamma(2u+1)\Gamma(v-q+1)}{\Gamma(2v+1)\Gamma(u-q+1)}M_{-q,v+\frac{1}{2}}(\gamma) \quad \text{if } v-u \geq 0,$$
$$= 0, \quad \text{if } v - u < 0. \tag{6.5}$$

Furthermore,

$$M_{m;0}^{v,u}(2\gamma) = I_m^{v,u}(\gamma), \tag{6.6}$$

where $I_m^{v,u}(\gamma)$ is the generalized Bessel function defined in I.

We list a few properties of the generalized Whittaker functions. The relations

$$\mathbf{T}(0,0,\gamma;\mathbf{e})f_m^{(u)} = \sum_{k=-\infty}^{\infty}\{u+k, m|\,0,0,\gamma,\mathbf{e}\,|u,m\}f_m^{(u+k)}$$

$$\{v, m|\,0,0,\gamma+\gamma';\mathbf{e}\,|u,m\} = \sum_{k=-\infty}^{\infty}\{v,m|\,0,0,\gamma;\mathbf{e}\,|u+k,m\} \times \{u+k,m|\,0,0,\gamma';\mathbf{e}\,|u,m\}$$

imply the identities

$$\frac{\Gamma(2m+l+1)l!}{\Gamma(m+l-q+1)}e^{\gamma z}P_l^{(m-q,m+q)}(z) = \sum_{k=0}^{\infty}\frac{k!\,\Gamma(k+2m+1)}{\Gamma(m+k-q+1)}M_{m;q}^{m+k,m+l}(2\gamma)P_k^{(m-q,m+q)}(z), \tag{6.7}$$

$$M_{m;q}^{u,v}(\gamma+\gamma') = \sum_{k=-\infty}^{\infty}M_{m;q}^{u+k,v}(\gamma)M_{m;q}^{u,u+k}(\gamma'), \tag{6.8}$$

convergent for all values of γ, γ'.

By applying the recursion relations (1.5′)–(1.8′) to expression (6.7), we can derive recursion relations for the generalized Whittaker functions:

$$\frac{(k+1)}{\gamma}M_{m;q}^{m+k+1,m+l}(\gamma) - \frac{l}{\gamma}M_{m+1;q}^{m+k+1,m+l}(\gamma)$$
$$= \frac{(m+k-q+1)}{(2m+2k+1)(2m+2k+2)}M_{m;q}^{m+k,m+l}(\gamma) - \frac{2(k+1)q}{(2m+2k+2)(2m+2k+4)}M_{m;q}^{m+k+1,m+l}(\gamma) - \frac{(k+1)(k+2)(m+q+k+2)}{(2m+2k+4)(2m+2k+5)}M_{m;q}^{m+k+2,m+l}(\gamma), \tag{6.9}$$

$$\frac{d}{d\gamma}M_{m;q}^{m+k,m+l}(\gamma)$$
$$= \frac{(m+l-q+1)}{2(m+l+1)(2m+2l+1)}M_{m;q}^{m+k,m+l+1}(\gamma) - \frac{mq}{2(m+l)(m+l+1)}M_{m;q}^{m+k,m+l}(\gamma) + \frac{l(l+m+q)(2m+l)}{(2m+2l+1)(2m+2l)}M_{m;q}^{m+k,m+l-1}(\gamma)$$

$$= \frac{(m+k-q)}{(2m+2k-1)(2m+2k)} M_{m;q}^{m+k-1,m+l}(\gamma) - \frac{mq}{2(m+k)(m+k+1)} M_{m;q}^{m+k,m+l}(\gamma) + \frac{(k+1)(k+2m+1)(k+m+q+1)}{(2m+2k+2)(2m+2k+3)} \times M_{m;q}^{m+k+1,m+l}(\gamma), \quad k, l = 0, 1, 2, \cdots.$$

The matrix elements $\{v, n|\ \alpha, 0, 0; \mathbf{e}\ |u, m\}$ can be determined easily from Lemma 4:

$$\{v, n|\ \alpha, 0, 0; \mathbf{e}\ |u, m\} = \frac{(2\alpha\omega)^{n-m}(u-m)!\,\Gamma(v-q+1)}{(n-m)!\,(u-m+n-v)!} \times \frac{\Gamma(u+m+v-n+1)}{\Gamma(m-q+v-n+1)\Gamma(2v+1)} \times {}_3F_2(v-n+m-u, u+m+v-n+1, v-q+1; m-q+v-n+1, 2v+2; 1), \quad \text{if } n-m \geq |v-u|,$$
$$= 0, \quad \text{otherwise.} \tag{6.10}$$

The addition theorem

$$\{v, n|\ \alpha+\alpha', 0, 0; \mathbf{e}\ |u, m\} = \sum_{u',m'} \{v, n|\ \alpha, 0, 0; \mathbf{e}\ |u', m'\}\{u', m'|\ \alpha', 0, 0; \mathbf{e}\ |u, m\}$$

leads to an identity for the functions ${}_3F_2(1)$ which the reader can derive for himself.

The matrix elements of the operators $\exp(\beta P^+)$ and $\exp(\beta P^-)$ are very similar. In fact, if we rewrite expression (1.8) in terms of basis vectors $f_m^{(u)\prime} = (-1)^u f_{-m}^{(u)}$, we see that Eqs. (1.7) and (1.8) become formally identical. Therefore, the matrix elements of $\exp(\beta P^-)$ can be obtained from the matrix elements (6.10) by formally replacing α, n, m by β, $-n$, $-m$, respectively, and multiplying the resulting expression by $(-1)^{v-u}$:

$$\{v, n|\ 0, \beta, 0; \mathbf{e}\ |u, m\} = \frac{(2\beta\omega)^{m-n}\Gamma(u+m+1)(-1)^{v-u}}{(m-n)!\,(u+m-n-v)!} \times \frac{\Gamma(u-m+v+n+1)\Gamma(v-q+1)}{\Gamma(n-q+v-m+1)\Gamma(2v+1)} \times {}_3F_2(v-m+n-u, u-m+v+n+1, v-q+1; n-q+v-m+1, 2v+2; 1), \quad \text{if } m-n \geq |v-u|,$$
$$= 0, \quad \text{otherwise.} \tag{6.11}$$

Another identity for the functions ${}_3F_2(1)$ can be derived from the addition theorem:

$$\sum_{u',m'} \{v, n|\ \alpha, 0, 0; \mathbf{e}\ |u', m'\}\{u', m'|\ 0, \beta, 0; \mathbf{e}\ |u, m\} = \sum_{u',m'} \{v, n|\ 0, \beta, 0; \mathbf{e}\ |u', m'\}\{u', m'|\ \alpha, 0, 0; \mathbf{e}\ |u, m\}.$$

In general, the matrix elements satisfy an addition theorem

$$\{v, n|\ \mathbf{w} + g\mathbf{w}'; gg'\ |u, m\} = \sum_{u',m'} \{v, n|\ \mathbf{w}, g\ |u', m'\}\{u', m'|\ \mathbf{w}', g'\ |u, m\}, \tag{6.12}$$

for g, g' in a sufficiently small neighborhood of $\mathbf{e} \in SL(2)$. This theorem can be used to derive identities relating the special functions constructed above. Several of these identities have been proved in I.

7. RELATIONS BETWEEN MATRIX ELEMENTS

We now investigate the relationship (for $g = \mathbf{e}$) between corresponding matrix elements of two different representations of $\mathcal{C}_6$, ρ, and ρ'. We suppose, first of all, that ρ and ρ' are distinct representations in the list (1)–(3), Sec. 1, except that they have the same ω. The representations may lie in different classes and may have different parameters $q, u_0; q', u_0'$. Denote the matrix elements of ρ by $\{v, n|\ \alpha, \beta, \gamma; \mathbf{e}\ |u, m\}$ and those of ρ' by $\{v', n'|\ \alpha, \beta, \gamma; \mathbf{e}\ |u', m'\}'$.

Making use of Model A and expression (3.2), we can express the ρ basis vectors as linear combinations of the ρ' basis vectors:

$$f_m^{(u)}(z, t) = \sum_{k=0}^{u-m} H(u, m, q; m', q', k) t^{m-m'} f_{m'}^{(m'+k)}(z, t)', \tag{7.1}$$

$$H(u, m, q; m', q'; k) = \frac{2^{m'-m}(u-m)!\,\Gamma(u+m+k+1)\Gamma(m'-q'+k+1)}{k!\,(u-m-k)!\,\Gamma(2m'+2k+1)\Gamma(m-q+k+1)} \times {}_3F_2(m-u+k, u+m+k+1, m'-q'+k+1; m-q+k+1, 2m'+2k+2; 1). \tag{7.2}$$

Application of the operator

$$\mathbf{T}(\alpha, \beta, \gamma; \mathbf{e}) = \exp[\omega(\alpha t + \beta(1-z^2)/t + \gamma z)]$$

to both sides of Eq. (7.1) yields the identity

$$\sum_{v,n} \{v, n|\ \alpha, \beta, \gamma; \mathbf{e}\ |u, m\} f_n^{(v)}(z, t) = t^{m-m'} \sum_{k=0}^{u-m} H(u, m, q; m', q'; k) \times \sum_{v',n'} \{v', n'|\ \alpha, \beta, \gamma; \mathbf{e}\ |m'+k, m'\}' f_{n'}^{(v')}(z, t)'.$$

The vectors $f_{n'}^{(v')}(z, t)'$ in this last expression can be expanded as linear combinations of vectors $f_n^{(v)}(z, t)$, where $n = m - m' + n'$ [use (7.1), interchanging primed and unprimed quantities]. Equating coefficients of $f_n^{(v)}(z, t)$ on both sides of the resulting identity, we find

$$\{v, n| \alpha, \beta, \gamma; \mathbf{e} |u, m\} = \frac{(u-m)!\,\Gamma(v-q+1)}{(v-n)!\,\Gamma(2v+1)\Gamma(v-m+m'-q'+1)}$$
$$\times \sum_{k=0}^{u-m} \sum_{l=v-m}^{\infty} \frac{\Gamma(u+m+k+1)\Gamma(m'-q'+k+1)}{k!\,(u-m-k)!\,\Gamma(2m'+2k+1)\Gamma(m-q+k+1)}$$
$$\times \frac{(m-n+l)!\,\Gamma(2m'+v-m+l+1)}{(m-v+l)!}$$
$$\times {}_3F_2(m-u+k, u+m+k+1, m'-q'+k+1; m-q+k+1, 2m'+2k+2; 1)$$
$$\times {}_3F_2(v-m-l, 2m'+v-m+l+1, v-q+1; m'+v-m-q'+1, 2v+2; 1)$$
$$\times \{m'+l, m'+n-m| \alpha, \beta, \gamma; \mathbf{e} |m'+k, m'\}', \; l, k, \text{ integers.} \qquad (7.3)$$

Equation (7.3) is a generalization of a number of important identities in special function theory. For example, if $\alpha = \beta = 0$; $n = m = u = v$, then this equation becomes

$$\gamma^{m'-v} M_{-q, v+\frac{1}{2}}(\gamma) = \sum_{l=0}^{\infty} \frac{(m'-q'+1)_l}{l!\,(2m'+l+1)_l} \times {}_3F_2(-l, 2m'+l+1, v-q+1; m'-q'+1, 2v+2; 1) M_{-q', m'+l+\frac{1}{2}}(\gamma), \qquad (7.4)$$

where

$$(\mu)_l = \Gamma(\mu + l)/\Gamma(\mu).$$

In case $q' = 0$, $m' = v - q$, identity (7.4) simplifies to

$$\frac{\gamma^{-q-\frac{1}{2}}}{\Gamma(2v+2)} M_{-q, v+\frac{1}{2}}(\gamma) = 2^{2v-2q+1}\Gamma(v-q+\tfrac{1}{2}) \times \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \frac{(v-q+l+\frac{1}{2})}{\Gamma(2v+l+2)} (2v-2q+1)_l \times (v-q)_l(-2q)_l I_{v-q+l+\frac{1}{2}}(\gamma/2). \qquad (7.5)$$

Next, the most general case, we will determine a relationship between the matrix elements of the representations ρ and ρ' when these representations correspond to different values of the nonzero parameter ω. The representation ρ has parameters q, u_0, ω, while ρ' has parameters q', u_0', ω'. There will be no loss of generality, if we assume $\omega' = 1$. As before, the matrix elements of ρ will be denoted by

$$\{v, n |\alpha, \beta, \gamma; \mathbf{e}| u, m\}$$

and those of ρ' by $\{v', n'| \alpha, \beta, \gamma; \mathbf{e} |u', m'\}'$.

If $\beta = \gamma = 0$, it is obvious from expression (6.10) that the matrix elements of ρ depend on ω according to the multiplicative factor ω^{n-m}. If $\alpha = \gamma = 0$, it follows from Eq. (6.11) that the matrix element varies as ω^{m-n}. However, if $\alpha = \beta = 0$, $\gamma \neq 0$, the ω dependence of the matrix elements is much more complicated.

To uncover the ω dependence we need a slight generalization of the identity (3.2):

Lemma 6: If $1 - x = \omega(1 - z)$, then

$$P_n^{(\gamma,\delta)}(z) = \sum_{k=0}^{n} \frac{\Gamma(\gamma+\delta+n+k+1)\Gamma(\alpha+\beta+k+1)\Gamma(\gamma+n+1)}{\Gamma(\alpha+\beta+2k+1)\Gamma(\gamma+\delta+n+1)\Gamma(\gamma+k+1)(n-k)!}$$
$$\times {}_3F_2(k-n, \gamma+\delta+n+k+1, \alpha+k+1; \gamma+k+1, \alpha+\beta+2k+2; \omega^{-1}) P_k^{(\alpha,\beta)}(x).$$

This lemma is proved in exactly the same way as the identity (3.2). Making use of Model A again, we observe that

$$\begin{aligned} \mathbf{T}(0, 0, \gamma; \mathbf{e}) f_m^{(u)}(z, t) &= e^{\omega\gamma z} f_m^{(u)}(z, t) \\ &= \exp[(\omega-1)\gamma + \gamma x] f_m^{(u)}(z, t) \\ &= \exp[(\omega-1)\gamma + \gamma x] \sum_{k=0}^{u-m} H^{\omega}(u, m, q; m', q'; k) \\ &\quad \times t^{m-m'} f_{m'}^{(m'+k)}(x, t)', \end{aligned} \qquad (7.6)$$

where $H^{\omega}(\cdot)$ is defined by Eq. (7.2), except that the function ${}_3F_2(1)$, occurring in Eq. (7.2), is replaced by ${}_3F_2(\omega^{-1})$. Thus,

$$\sum_{v,n} \{v, n| 0, 0, \gamma; \mathbf{e} |u, m\} f_n^{(v)}(z, t) = e^{(\omega-1)\gamma} t^{m-m'} \sum_{k=0}^{u-m} H^{\omega}(u, m, q; m', q'; k) \times \sum_{v',n'} \{v', n'| 0, 0, \gamma; \mathbf{e} |m'+k, m'\}' f_{n'}^{(v')}(x, t)'.$$

Expanding the right-hand side of this expression in terms of the basis $f_n^{(v)}(z, t)$, $n = m - m' + n'$, and equating coefficients of the basis vectors, we obtain the identity

$$M^{v,u}_{m;q}(\gamma\omega) = \frac{(u-m)!\,\Gamma(v-q+1)\exp[\frac{1}{2}(\omega-1)\gamma]}{(v-m)!\,\Gamma(2v+1)\Gamma(v-m+m'-q'+1)} \times \sum_{k=0}^{u-m}\sum_{l=v-m}^{\infty} \frac{\Gamma(u+m+k+1)\Gamma(m'-q'+k+1)l!\,\Gamma(2m'+v-m+l+1)}{k!\,(u-m-k)!\,\Gamma(2m'+2k+1)\Gamma(m-q+k+1)(m-v+l)!}$$
$$\times {}_3F_2(m-u+k, u+m+k+1, m'-q'+k+1; m-q+k+1, 2m'+2k+2; \omega^{-1})$$
$$\times {}_3F_2(v-m-l, 2m'+v-m+l+1, v-q+1; m'+v-m-q'+1, 2v+2; \omega) M^{m'+l,m'+k}_{m';q'}(\gamma). \quad (7.7)$$

If $m = u = v$, this identity becomes

$$\gamma^{m'-v}M_{-q,v+\frac{1}{2}}(\gamma\omega) = \exp[\tfrac{1}{2}(\omega-1)\gamma]\omega^{v+1}\sum_{l=0}^{\infty}\frac{(m'-q'+1)_l}{l!\,(2m'+l+1)_l} \times {}_3F_2(-l, 2m'+l+1, v-g+1; m'-q'+1, 2v+2; \omega)M_{-q',m'+l+\frac{1}{2}}(\gamma). \quad (7.8)$$

When $v - q = m' - q'$, Eq. (7.8) simplifies to

$$\gamma^{m'-v}M_{-q'+m'-v,v+\frac{1}{2}}\left(\gamma\frac{1-\xi}{2}\right) = \exp\left[-\frac{\gamma(1+\xi)}{4}\right]\left(\frac{1-\xi}{2}\right)^{v+1} \times \sum_{l=0}^{\infty}\frac{(m'-q'+1)_l}{(2v+2)_l}\frac{1}{(2m'+l+1)_l} \times P_l^{(2v+1,2m'-2v-1)}(\xi)M_{-q',m'+l+\frac{1}{2}}(\gamma), \quad (7.9)$$

where $\xi = 2\omega - 1$.

8. MODELS IN THREE COMPLEX VARIABLES

It was shown in I that the representations $\uparrow_4(\omega, 0)$, $\uparrow_3(\omega, 0)$, and $R_3(\omega, 0, u_0)$ have models in terms of differential operators and analytic functions in three complex variables. Those representations for which $q \neq 0$, however, have no such models. On the other hand, we will show that the matrix elements

$$\{v, n|\ \alpha, \beta, \gamma, \mathbf{e}\ |u, m\}$$

of the representations $\uparrow_4(\omega, q)$, $\uparrow_3(\omega, q)$, and $R_3(\omega, q, u_0)$ themselves define models in terms of differential operators acting on vector-valued functions of three complex variables. To see this, we consider a representation ρ from one of the classes listed above and note the relation

$$\{\mathbf{w}, g\} = \{\mathbf{0}, g\}\{g^{-1}\mathbf{w}, \mathbf{e}\} = \{\mathbf{w}, \mathbf{e}\}\{\mathbf{0}, g\},$$

which leads to the addition theorem

$$\sum_{n'} U^v_{n,n'}(g)\{v, n'|\ g^{-1}\mathbf{w}, \mathbf{e}\ |u, m\} = \sum_{m'}\{v, n|\ \mathbf{w}, \mathbf{e}\ |u, m'\}U^u_{m',m}(g) \quad (8.1)$$

for the matrix elements of ρ. Here,

$$U^v_{n,n'}(g) = \{v, n|\ \mathbf{0}, g\ |v, n'\}$$

and g is in a small enough neighborhood of $\mathbf{e} \in SL(2)$ so that all terms in Eq. (8.1) make sense.

Fix v, and consider the vector-valued function

$$\mathrm{X}^\rho_{v;u,m}(\mathbf{w}) = (\{v, n|\ \mathbf{w}, \mathbf{e}\ |u, m\}). \quad (8.2)$$

Here, n runs over the values $n = -v, -v+1, \cdots, +v$, if $\rho = \uparrow_4(\omega, q)$ and $n = v, v-1, v-2, \cdots$ if $\rho = \uparrow_3(\omega, q)$ or $\rho = R_3(\omega, q, u_0)$. Define the action $\mathbf{T}$ of T_6 on $\mathrm{X}(\mathbf{w})$ by (in matrix notation)

$$[\mathbf{T}(\mathbf{a}, g)\mathrm{X}^\rho_{v;u,m}](\mathbf{w}) = U^v(g)\mathrm{X}^\rho_{v;u,m}(g^{-1}(\mathbf{w}+\mathbf{a})). \quad (8.3)$$

Clearly,

$$\mathbf{T}(g\mathbf{a}' + \mathbf{a}, gg') = \mathbf{T}(\mathbf{a}, g)\mathbf{T}(\mathbf{a}', g').$$

According to Eq. (8.1), the vector-valued function $\mathrm{X}^\rho_{v;u,m}(\mathbf{w})$ transforms like the basis vector $f^{(u)}_m$ under the operator $\mathbf{T}(\mathbf{0}, g)$. Furthermore, it is easy to verify the relation

$$[\mathbf{T}(\mathbf{a}, \mathbf{e})\mathrm{X}^\rho_{v;u,m}](\mathbf{w}) = \sum_{v',n'}\{v', n'|\ \mathbf{a}, \mathbf{e}\ |u, m\}\mathrm{X}^\rho_{v;v',n'}(\mathbf{w}). \quad (8.4)$$

It follows from these expressions that the operators $\mathbf{T}(\mathbf{a}, g)$ and the vectors $\mathrm{X}^\rho_{v;u,m}(\mathbf{w}) \equiv f^{(u)}_m$ define a model (Model C) of the abstract representation ρ. Standard methods in the theory of Lie transformation groups[4,7] can be used to compute the infinitesimal operators corresponding to this model. The results are

$$\begin{aligned} J^+ &= \gamma\frac{\partial}{\partial\alpha} + 2\beta\frac{\partial}{\partial\gamma} + S^+, \\ J^- &= -\gamma\frac{\partial}{\partial\beta} + 2\alpha\frac{\partial}{\partial\gamma} + S^-, \\ J^3 &= -\alpha\frac{\partial}{\partial\alpha} + \beta\frac{\partial}{\partial\beta} + S^3, \\ P^+ &= \frac{\partial}{\partial\alpha}, \quad P^- = \frac{\partial}{\partial\beta}, \quad P^3 = \frac{\partial}{\partial\gamma}, \end{aligned} \quad (8.5)$$

where

$$S^\pm\{v, n\ |\cdot|\ u, m\} = (\pm n - v)\{v, n \mp 1\ |\cdot|\ u, m\},$$
$$S^3\{v, n\ |\cdot|\ u, m\} = n\{v, n\ |\cdot|\ u, m\}.$$

It is an immediate consequence of these results that the vectors $\mathrm{X}^\rho_{v;u,m}(\mathbf{w}) \equiv f^{(u)}_m$ and the infinitesimal operators (8.5) satisfy the recursion relations (1.5)–(1.8). Lemmas 1–5 can now be used to provide additional information about the Model C basis vectors. For example, Corollary 2 yields the identity

$$P_l^{(m-q,m+q)}\left(\omega^{-1}\frac{\partial}{\partial\gamma}\right)\mathrm{X}^\rho_{v;m,m}(\mathbf{w}) = \frac{\Gamma(2m+1)\Gamma(m-q+l+1)}{l!\,\Gamma(2m+l+1)\Gamma(m-q+1)}\mathrm{X}^\rho_{v;m+l,m}(\mathbf{w}). \quad (8.6)$$

This identity, as well as all others obtained from Lemmas 1–5, constitute generalizations of the "Maxwell theory of poles" for solutions of the wave equation.[12]

[12] A. Erdelyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill Book Co., Inc., New York, 1953), Vol. 2, Chap. 11.

Degenerate Representations of the Symplectic Groups. I. The Compact Group $Sp(n)$

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The degenerate, irreducible, unitary representations of the compact group $Sp(n)$, characterized by one and two invariant numbers, are considered. The explicit expressions for the basis functions spanning the corresponding representation spaces and the decomposition with respect to the maximal subgroup are given.

1. INTRODUCTION

Many attempts have been made in the last few years to understand the properties of physical systems such as elementary particles, the hydrogen atom, nuclei, etc., using the theory of representations of the underlying symmetry group. The main effort was devoted to the rotation and unitary groups, while the class of symplectic groups did not receive much attention. This may be due to the peculiar property of these groups, that of conserving an antisymmetric, bilinear form.

Some interest in symplectic groups was raised by remarks of Lipkin[1] on possible applications of the group $Sp(n, R)$ to systems of bosons which do not conserve the number of particles. Budini[2] has pointed out that, using $Sp(6, 6)$ as a higher symmetry group, it is possible to obtain a mass formula for elementary particles without symmetry breaking. The questions of the symplectic symmetry of hadrons and of the embedding of the harmonic oscillator in the symplectic group have been discussed in Ref. 3. On the other hand, the theory of the degenerate representations of the rotation and unitary (both compact and noncompact) groups has been developed in a series of papers.[4] In this work we present the extension of that approach to the unitary symplectic groups, i.e., those which conserve both symmetric and antisymmetric bilinear forms.

In general, the irreducible unitary representations of a semisimple Lie group G are realized as mappings of a Hilbert space $\mathcal{H}(X)$ into itself, the domain of corresponding functions being some homogeneous space X of the type

$$X = G/G_0, \tag{1.1}$$

when G_0 is a closed subgroup of G.

Gel'fand[5] has proved the important theorem which states that the number of independent invariant operators in the enveloping algebra acting in the Hilbert space of functions $\mathcal{H}(X)$ with domain X[6] is equal to the rank[7] of the space X (and is therefore independent of the rank of the fundamental group G). Since we are primarily interested in construction of representations characterized by the minimum number of invariants, we can use this theorem to select an appropriate domain X, namely, that of rank one.

In order to select the proper invariant operator we can use the theorem of Helgason,[8] according to which the ring of invariant operators in the algebra $\mathfrak{R}$ of the group G, realized on the space of rank one, is generated by the Laplace–Beltrami operator

$$\Delta(X) = \frac{1}{(|\bar{g}|)^{\frac{1}{2}}} \partial_\alpha g^{\alpha\beta}(X)(|\bar{g}|)^{\frac{1}{2}}\partial_\beta \tag{1.2}$$

on X. Here $g^{\alpha\beta}(X)$ is defined by

$$g^{\alpha\beta}(X)g_{\beta\gamma}(X) = \delta^\alpha_\gamma, \tag{1.3}$$

where $g_{\alpha\beta}(X)$ is the metric tensor on the space X and $|\bar{g}| = |\det\{g_{\alpha\beta}(X)\}|$.

Then the problem of construction of the most degenerate, irreducible, unitary representations is reduced to the problem of determining eigenfunctions and eigenvalues of the Laplace–Beltrami operator on the appropriate symmetric space $X = G/G_0$ of rank one.

We select a suitable domain X and solve the eigenproblem of the Laplace–Beltrami operator on it in

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[1] H. J. Lipkin, *Lie Group for Pedestrians* (North-Holland Publishing Company, Amsterdam, 1965).

[2] P. Budini, private communication.

[3] H. Bacry, J. Nuyts, and L. Van Hove, Nuovo Cimento **35**, 510 (1965); R. Hwa and J. Nuyts, Phys. Rev. **145**, 1188 (1966).

[4] N. Limić, J. Niederle, and R. Rączka, J. Math. Phys. **7**, 1861, 2026 (1966); J. Fischer and R. Rączka, Commun. Math. Phys, **3**, 233 (1966); **4**, 8 (1967).

[5] I. M. Gel'fand, Am. Math. Soc. Transl. Ser. 2, **37**, 31 (1964).

[6] Note that sometimes in the set of invariant operators there appear operators which are not elements of the enveloping algebra. (For an explicit example see Ref. 4.)

[7] The rank of a space $X = G/G_0$ is defined as the number of invariants of any two points $x, y \in X$ with respect to the action of the fundamental group G on X.

[8] S. Helgason, *Differential Geometry and Symmetric Spaces* (Academic Press Inc., New York, 1962), Chap. 4, p. 397.

TABLE I. Homogeneous spaces connected with symplectic groups.

Cartan's list: G_0 compact $X = \frac{G}{G_0}$	Rank of X	Dimension of X	Rozenfeld's list: G_0 noncompact $X = \frac{G}{G_0}$	Rank of X	Dimension of X
$\frac{Sp(n)}{\mathfrak{U}(n)}$	n	$n(n+1)$	$\frac{Sp(p,q)}{\mathfrak{U}(p,q)}$	$p+q$	$(p+q)(p+q+1)$
$\frac{Sp(n,\mathbb{R})}{\mathfrak{U}(n)}$	n	$n(n+1)$	$\frac{Sp(p+q,\mathbb{R})}{\mathfrak{U}(p,q)}$	$p+q$	$(p+q)(p+q+1)$
$\frac{Sp(p+q)}{Sp(p)\otimes Sp(q)}$	$\min(p,q)$	$4pq$	$\frac{Sp(p+q,\mathbb{R})}{Sp(p,\mathbb{R})\otimes Sp(q,\mathbb{R})}$	$\min(p,q)$	$4pq$
$\frac{Sp(p,q)}{Sp(p)\otimes Sp(q)}$	$\min(p,q)$	$4pq$	$\frac{Sp(p,q)}{Sp(k,m)\otimes Sp(p-k,q-m)}$	$\min[(k+m), (p+q-k-m)]$	$4(p+q-k-m) \times (k+m)$

Sec. 2. Section 3 is then devoted to the study of the properties of the most degenerate representation of the group $Sp(n)$ obtained in this way. In Sec. 4 we discuss some aspects of the determination of the series of less degenerate representations of $Sp(n)$ characterized by two independent numbers. Thus in the present paper we deal only with the case of the compact group $Sp(n)$. We shall, however, use the results obtained here in forthcoming papers, in which we would like to solve the following problems:

(i) The construction of a representation space for (most) degenerate irreducible unitary representations of the noncompact, unitary, symplectic group $Sp(p, q)$ determined by a discrete or a continuous invariant;

(ii) The decomposition of the tensor product of two representations of $Sp(p, q)$ group into irreducible components and the decomposition of the irreducible unitary representations of $Sp(p, q)$ with respect to compact and/or noncompact subgroups.

2. CONSTRUCTION OF THE REPRESENTATION SPACE

According to Gel'fand's theorem,[5] the properties of the irreducible unitary representations of a group G realized on a Hilbert space $\mathcal{H}(X)$ are determined by the geometrical properties of a domain X of functions $f(X) \in \mathcal{H}(X)$, the domain X being some homogeneous space.

Symmetric spaces of the type (1.1) with a compact stability group G_0 have been classified by Cartan, whereas those with noncompact stability group have been listed by Rozenfeld.[9] We reproduce in Table I Cartan's list of symmetric spaces (see Ref. 8) for the fundamental group G of the symplectic type. There also is collected the spaces from Rozenfeld's list, together with their ranks and dimensions.

[9] B. A. Rozenfeld, Dokl. Akad. Nauk SSSR **110**, 23 (1956).

We see that the only suitable candidate for a space of rank one on which the compact group $Sp(n)$ acts transitively is the space

$$X_1 = \frac{Sp(n)}{Sp(n-1)\otimes Sp(1)}. \tag{2.1}$$

This space is known to be a quaternionic projective space.[8,9] But it is rather difficult to construct a convenient and simple geometrical model for it. Fortunately, we may use for our purposes the space

$$X^{4n-1} = \frac{Sp(n)\otimes Sp(1)}{Sp(n-1)\otimes Sp(1)} \approx \frac{Sp(n)}{Sp(n-1)}, \tag{2.2}$$

which is evidently closely related to the space (2.1). Furthermore, the space X^{4n-1} is isomorphic to the unitary sphere in the n-dimensional quaternionic unitary space $\mathfrak{Q}^{(n)}$, defined by the equation

$$\sum_{k=1}^{n} \bar{q}_k q_k = 1. \tag{2.3}$$

It has been proved by Chevalley[10] and Hsien-Chung[11] that the group $Sp(n)\otimes Sp(1)$ acts transitively on (2.3) and that its stability group is $Sp(n-1)\otimes Sp(1)$.

As is well known, the noncommutative algebra of quaternions $\mathfrak{Q}$ is defined as an algebra of dimension 4 over the field $\mathbb{R}$ of real numbers with a base composed of four elements $1, i, j, k$ whose multiplication table is

$$\begin{array}{c|cccc} & 1 & i & j & k \\ \hline 1 & 1 & i & j & k \\ i & i & -1 & k & -j \\ j & j & -k & -1 & i \\ k & k & j & -i & -1. \end{array} \tag{2.4}$$

[10] C. Chevalley, *Theory of Lie Groups* (Princeton University Press, Princeton, N.J., 1964), Vol. I.

[11] Wang Hsien-Chung, Ann. Math. **55**, 177 (1952).

Then any quaternion $q \in \mathcal{Q}$ may be expressed either in the form

$$q = x_1 + x_2 i + x_3 j + x_4 k, \qquad (2.5)$$

where x_i $(i = 1, \cdots, 4)$ are real numbers, or

$$q = z_1 + z_2 j, \qquad (2.5')$$

where z_i $(i = 1, 2)$ are complex numbers. The quaternionic conjugation is the mapping

$$q \to \bar{q} = x_1 - x_2 i - x_3 j - x_4 k = z_1^* - z_2 j \qquad (2.6)$$

of $\mathcal{Q}$ onto itself. (For a detailed treatment of properties of the body of quaternions, as well as for the questions about the relation of symplectic groups to the vector spaces over the body of quaternions, see, for example, the book of Chevalley.[10])

It is important that the n-dimensional quaternionic unitary sphere (2.3) is homeomorphic to the usual sphere in $4n$-dimensional Euclidean space $\mathbb{R}^{4n}$. Because of it, its properties are rather simple.

Now let us introduce an inner coordinate system on the sphere (2.3). Let us suppose that we have defined a coordinate system on the quaternionic unitary sphere (2.3) of dimension $p < n$. Let these coordinates be denoted by q'_κ $(k = 1, \cdots, p)$. Then the coordinate system on the "sphere" of dimension $p + 1$ will be defined by

$$q_k = q'_k \sin \xi_{p+1} \quad \text{for} \quad k = 1, \cdots, p, \qquad (2.7)$$

$$q_{p+1} = (e^{i\varphi_{p+1}} \cos \vartheta_{p+1} + e^{i\psi_{p+1}} \sin \vartheta_{p+1} j) \cos \xi_{p+1}. \qquad (2.8)$$

Now starting from

$$q_1 = (e^{i\varphi_1} \cos \vartheta_1 + e^{i\psi_1} \sin \vartheta_1 j), \qquad (2.9)$$

we get the coordinate system for an arbitrary dimension of the quaternionic unitary sphere (2.3) using recursive formulas (2.7) and (2.8). This choice is convenient because there appears [in parentheses in (2.8) and (2.9)] a general expression for a quaternion of modulus equal to one. The ranges of variables φ_k, ψ_k, ϑ_k, and ξ_k must be chosen so that the coordinates (2.7) cover the space X^{4n-1} only once. In this way, on the space X^{4n-1} we introduce

n variables $\varphi_k \in [0, 2\pi)$ $(k = 1, \cdots, n)$,
n variables $\psi_k \in [0, 2\pi)$ $(k = 1, \cdots, n)$,
n variables $\vartheta_k \in [0, \pi/2]$ $(k = 1, \cdots, n)$,

and

$(n - 1)$ variables $\xi_k \in [0, \pi/2]$ $(k = 2, \cdots, n)$, (2.10)

i.e., $4n - 1$ variables altogether.

The metric tensor $g_{\alpha\beta}(X^{4n-1})$, induced by the metric tensor of the quaternionic unitary space $\mathcal{Q}^{(n)}$, is given by the symmetric part of the tensor $g'_{\alpha\beta}$ defined by

$$g'_{\alpha\beta}(X^{4n-1}) = \sum_{s,t=1}^{n} g_{st}(\mathcal{Q}^{(n)}) \frac{\partial \bar{q}_s}{\partial \Omega_\alpha} \frac{\partial q_t}{\partial \Omega_\beta}, \qquad (2.11)$$

where

$$\alpha = (s, \sigma), \quad \beta = (t, \tau)$$

and

$$\Omega_{(s,1)} = \varphi_s, \quad \Omega_{(s,2)} = \psi_s, \quad \Omega_{(s,3)} = \vartheta_s, \quad \Omega_{(s,4)} = \xi_s.$$

In our parametrization the metric tensor $g_{\alpha\beta}(X^{4n-1})$ is diagonal, and therefore the Laplace–Beltrami operator (1.2) can be represented in the form

$$\Delta(X^{4n-1}) = \frac{1}{(\cos \xi_n)^3 (\sin \xi_n)^{4n-5}} \frac{\partial}{\partial \xi_n} (\cos \xi_n)^3 (\sin \xi_n)^{4n-5} \frac{\partial}{\partial \xi_n} + \frac{1}{(\cos \xi_n)^2} \hat{K}_n + \frac{1}{(\sin \xi_n)^2} \Delta(X^{4(n-1)-1}), \qquad (2.12)$$

where

$$\hat{K}_n = \frac{1}{\sin \vartheta_n \cos \vartheta_n} \frac{\partial}{\partial \vartheta_n} \sin \vartheta_n \cos \vartheta_n \frac{\partial}{\partial \vartheta_n} + \frac{1}{(\cos \vartheta_n)^2} \frac{\partial^2}{\partial \varphi_n^2} + \frac{1}{(\sin \vartheta_n)^2} \frac{\partial^2}{\partial \psi_n^2} \qquad (2.13)$$

and $\Delta(X^{4(n-1)-1})$ is the Laplace–Beltrami operator on the quaternionic unitary sphere embedded in the space $\mathcal{Q}^{(n-1)}$. For $n = 1$ we have

$$\Delta(X^3) = \hat{K}_1. \qquad (2.14)$$

To find the basis functions for the Hilbert space $\mathcal{H}(X)$ on which the representations of the $Sp(n)$ group may be realized, we have to solve the equation

$$\Delta(X^{4n-1}) V_{\lambda_{(n)}}(\Omega^{(n)}) = \lambda_{(n)} V_{\lambda_{(n)}}(\Omega^{(n)}), \qquad (2.15)$$

where $\Omega^{(n)}$ stands for the set of variables $\{\Omega_1, \cdots, \Omega_n\}$. Representing solutions $V_{\lambda_{(n)}}(\Omega^{(n)})$ of (2.15) in the form

$$V_{\lambda_{(n)}}(\Omega^{(n)}) = \Phi_n(\varphi_n) \Psi_n(\psi_n) \Theta_n(\vartheta_n) \times \Xi_n(\xi_n) V_{\lambda_{(n-1)}}(\Omega^{(n-1)}), \qquad (2.16)$$

we obtain the set of ordinary differential equations of second order:

$$\frac{d^2 \Phi_n}{d\varphi_n^2} + m_n^2 \Phi_n = 0, \qquad (2.17)$$

$$\frac{d^2 \Psi_n}{d^2 \psi_n^2} + \bar{m}_n^2 \Psi_n = 0, \qquad (2.18)$$

$$\left[\frac{1}{\sin \vartheta_n \cos \vartheta_n} \frac{d}{d\vartheta_n} \sin \vartheta_n \cos \vartheta_n \frac{d}{d\vartheta_n} + \kappa_n - \frac{m_n^2}{(\cos \vartheta_n)^2} - \frac{\bar{m}_n^2}{(\sin \vartheta_n)^2}\right] \Theta_n(\vartheta_n) = 0, \qquad (2.19)$$

$$\left[\frac{1}{(\cos \xi_n)^3 (\sin \xi_n)^{4n-5}} \frac{d}{d\xi_n} (\cos \xi_n)^3 (\sin \xi_n)^{4n-5} \frac{d}{d\xi_n} - \lambda_{(n)} + \frac{\lambda_{(n-1)}}{(\sin \xi_n)^2} - \frac{\kappa_n}{(\cos \xi_n)^2}\right] \Xi_n(\xi_n) = 0, \qquad (2.20)$$

$$\Delta(X^{4(n-1)-1}) V_{\lambda_{(n-1)}}(\Omega^{(n-1)}) = \lambda_{(n-1)} V_{\lambda_{(n-1)}}(\Omega^{(n-1)}). \qquad (2.21)$$

General solutions of Eqs. (2.19) and (2.20) are given in terms of hypergeometrical functions as follows:

$$\Theta_n(\vartheta_n) = (\tan\vartheta_n)^{|\bar{m}_n|}(\cos\vartheta_n)^{l_n} \times {}_2F_1\left(\frac{|\bar{m}_n| - l_n + |m_n|}{2}, \frac{|\bar{m}_n| - l_n - |m_n|}{2}; |\bar{m}_n| + 1; -\tan^2\vartheta_n\right), \quad (2.22)$$

$$\Xi_n(\xi_n) = (\tan\xi_n)^{L_{n-1}}(\cos\xi_n)^{L_n} \times {}_2F_1\left(\frac{L_{n-1} - L_n + l_n}{2}, \frac{L_{n-1} - L_n - l_n - 2}{2}; L_{n-1} + 2(n-1); -\tan^2\xi_n\right), \quad (2.23)$$

where the eigenvalues m_n and $\bar{m}_n$ are integers and the spectrum of remaining eigenvalues is given by

$$\kappa_n = l_n(l_n + 2) \quad (2.24)$$

and

$$\lambda_{(n)} = -L_n(L_n + 4n - 2) \quad (2.25)$$

with positive integers l_n and L_n.

These solutions are square-integrable with respect to the measure $d\mu(X)$ if the following restrictions on eigenvalues are satisfied[12]:

$$|\bar{m}_n| + |m_n| = l_n - 2k, \quad k = 0, 1, \cdots, [l_n/2] \quad (2.26)$$

and

$$L_{n-1} + l_n = L_n - 2k', \quad k' = 0, 1, \cdots, [L_n/2]. \quad (2.27)$$

The solutions (2.22) and (2.23) are expressible in terms of the usual d functions of the theory of angular momenta.[13] Then the eigenfunctions of (2.15) are explicitly given by

$$\begin{aligned} &Y^{L_n, L_{n-1}, \cdots, L_2; l_n, \cdots, l_1;}_{m_n, \cdots, m_1; \bar{m}_n, \cdots, \bar{m}_1;}(\Omega^{(n)}) \\ &\equiv Y^{L, L_i; l_i}_{m_i, \bar{m}_i}(\Omega^{(n)}) \\ &= (N_n)^{-\frac{1}{2}} e^{i(m_1\varphi_1 + \bar{m}_1\psi_1)} d^{j_1}_{a_1, b_1}(2\vartheta_1) \\ &\times \prod_{k=2}^{n} e^{i(m_k\varphi_k + m_k\psi_k)} d^{j_k}_{a_k, b_k}(2\vartheta_k) \frac{(\sin\xi_k)^{3-2k}}{\cos\xi_k} d^{J_k}_{\alpha_k, \beta_k}(2\xi_k), \end{aligned} \quad (2.28)$$

where

$$\begin{aligned} a_k &= \tfrac{1}{2}(|m_k| - |\bar{m}_k|), \quad b_k = \tfrac{1}{2}(|m_k| + |\bar{m}_k|), \\ \alpha_k &= \tfrac{1}{2}(l_k - L_{k-1} + 4 - 2k), \\ \beta_k &= \tfrac{1}{2}(l_k + L_{k-1} + 2k - 2), \end{aligned} \quad (2.29)$$

and

$$J_k = \tfrac{1}{2}(L_k + 2k - 2), \quad j_k = l_k/2. \quad (2.30)$$

[12] Here $[a]$ means the integral part of a defined as usual.

[13] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1961).

The normalization constant N_n is then

$$N_n = \pi^{2n}(l_1 + 1)\prod_{k=2}^{n}(l_k + 1)(L_k + 2k - 1). \quad (2.31)$$

The functions (2.28) with a given value $L \equiv L_n$ are square-integrable with respect to the left-invariant measure

$$d\mu(\Omega^{(n)}) = [|\bar{g}(X^{4n-1})|]^{\frac{1}{2}} d\Omega^{(n)} \quad (2.32)$$

on the domain X^{4n-1}. The explicit expression for the measure $d\mu$ is

$$\begin{aligned} d\mu(\Omega^{(n)}) &= \cos\vartheta_1 \sin\vartheta_1 \, d\varphi_1 \, d\psi_1 \, d\vartheta_1 \\ &\times \prod_{k=2}^{n} \cos\vartheta_k \sin\vartheta_k (\cos\xi_k)^3 (\sin\xi_k)^{4k-5} \\ &\times d\varphi_k \, d\psi_k \, d\vartheta_k \, d\xi_k. \end{aligned} \quad (2.33)$$

Hence, the set of functions $Y^{L, L_i; l_i}_{m_i, \bar{m}_i}(\Omega^{(n)})$ span the Hilbert space $\mathcal{H}^L(X^{4n-1})$ with the scalar product related to the left-invariant measure (2.32) by

$$(\eta, \chi) = \int_{X^{4n-1}} \overline{\eta(\Omega^{(n)})}\, \chi(\Omega^{(n)}) \, d\mu(\Omega^{(n)}), \quad (2.34)$$

for any $\eta, \chi \in \mathcal{H}^L(X^{4n-1})$. In fact, the space $\mathcal{H}^L(X^{4n-1})$ is a representation space of the group $Sp(1) \otimes Sp(n)$, which occurs as a fundamental group of the space (2.2). However, a closer study of the properties of the Lie algebras of $Sp(1)$ and $Sp(n)$ groups reveals that we can realize irreducible unitary representations of $Sp(n)$ on certain subspaces of the Hilbert space $\mathcal{H}^L(X^{4n-1})$.

3. MOST DEGENERATE REPRESENTATIONS OF $Sp(n)$

A. Structure of the Lie Algebra

The group $Sp(1) \otimes Sp(n)$ which acts on the manifold (2.2) is a direct product of two simple groups. Therefore its Lie algebra $\mathfrak{R}$ decomposes into two commuting subalgebras which we call $\mathfrak{R}_1'$ and $\mathfrak{R}_n$, respectively. The algebra $\mathfrak{R}_n$ of $Sp(n)$ is then formed by the $n(2n+1)$ generators

$$\mathcal{U}^+_{i,j}, \mathcal{U}^-_{i,j}, \mathcal{V}^+_{i,j}, \text{ and } \mathcal{V}^-_{i,j}, \quad i, j = 1, \cdots, n, \quad (3.1)$$

which have the symmetry properties

$$\mathcal{U}^{\pm}_{i,j} = \mathcal{U}^{\pm}_{j,i}, \quad \mathcal{V}^+_{i,j} = \mathcal{V}^+_{j,i}, \quad \text{and} \quad \mathcal{V}^-_{i,j} = -\mathcal{V}^-_{j,i}. \quad (3.2)$$

The commutation relations of these generators are

$$[\mathcal{U}^{\overset{+}{\pm}}_{i,j}, \mathcal{U}^{\underset{-}{\pm}}_{k,l}] = \overset{-}{\mp}\tfrac{1}{2}\{\delta_{jk}\mathcal{V}^{\overset{-}{\pm}}_{i,l} + \delta_{jl}\mathcal{V}^{\overset{-}{\pm}}_{i,k} + \delta_{ik}\mathcal{V}^{\overset{-}{\pm}}_{j,l} + \delta_{il}\mathcal{V}^{\overset{-}{\pm}}_{j,k}\}, \quad (3.3)$$

$$[\mathcal{V}^{\overset{+}{\pm}}_{i,j}, \mathcal{V}^{\underset{-}{\pm}}_{k,l}] = \underset{+}{\mp}\tfrac{1}{2}\{\delta_{jk}\mathcal{V}^{\overset{-}{\pm}}_{i,l} \pm \delta_{jl}\mathcal{V}^{\overset{-}{\pm}}_{i,k} \overset{+}{\pm} \delta_{ik}\mathcal{V}^{\overset{-}{\pm}}_{jl} \underset{+}{\pm} \delta_{il}\mathcal{V}^{\overset{-}{\pm}}_{j,k}\}, \quad (3.4)$$

$$[\mathcal{U}^{\overset{+}{\pm}}_{i,j}, \mathcal{V}^{\overset{\pm}{\mp}}_{k,l}] = \overset{\mp}{\underset{+}{+}}\tfrac{1}{2}\{\delta_{jk}\mathcal{U}^{\overset{\mp}{\mp}}_{i,l} \overset{\pm}{\mp} \delta_{jl}\mathcal{U}^{\overset{\mp}{\mp}}_{i,k} + \delta_{ik}\mathcal{U}^{\overset{\mp}{\mp}}_{j,l} \overset{\pm}{\mp} \delta_{il}\mathcal{U}^{\overset{\mp}{\mp}}_{j,k}\}. \quad (3.5)$$

In Appendix A we have collected the explicit expressions for the generators (3.1) as linear differential operators in quaternionic and complex variables as well as their connection with the generators of the group $\mathcal{U}(2n)$.

The algebra $\mathcal{R}_1'$ of $Sp(1)$ is generated by the three operators $\mathcal{W}_l$ $(l = 1, 2, 3)$ which, on the manifold X^{4n-1}, have the form

$$\mathcal{W}_1 = \sum_{k=1}^{n} \tilde{\mathcal{U}}_{k,k}^{+}, \quad \mathcal{W}_2 = \sum_{k=1}^{n} \tilde{\mathcal{U}}_{k,k}^{-}, \quad \mathcal{W}_3 = \sum_{k=1}^{n} \tilde{\mathcal{V}}_{k,k}^{+}. \quad (3.6)$$

There is a close relation between the operators $\tilde{\mathcal{U}}_{kk}^{\pm}$, $\tilde{\mathcal{V}}_{kk}^{+}$, and $\mathcal{U}_{kk}^{\pm}$, $\mathcal{V}_{kk}^{+}$. It is easiest to see from expressions (A11) and (A18) of Appendix A, which define the generators in terms of complex variables. The meaning of the tilde in Eqs. (3.6) then consists in the substitution $z_k \leftrightarrows z_k^*$ for only $k = -1, \cdots, -n$, while remaining variables are unchanged ($z_k \to z_k$ and $z_k^* \to z_k^*$ for $k = 1, \cdots, n$).

The commutation relations for the generators (3.6) are

$$[\mathcal{W}_i, \mathcal{W}_j] = 2\epsilon_{ijk}\mathcal{W}_k \quad (k = 1, 2, 3). \quad (3.7)$$

Throughout this paper we often use, instead of (3.1) and (3.6), the set of generators of the complex extension of the real Lie algebras $\mathcal{R}_n$ and $\mathcal{R}_1$. These are especially convenient when dealing with the basis functions (2.28) and can be normalized in such a way that they form the Weyl's standard basis. We define these operators by

$$E_{\pm 2e_k} = 2^{-\frac{1}{2}}(\mathcal{U}_{k,k}^{+} \pm i\mathcal{U}_{k,k}^{-}), \quad (3.8)$$

$$E_{\pm e_k \pm e_l} = (\mathcal{U}_{k,l}^{+} \pm i\mathcal{U}_{k,l}^{-}), \quad (3.9)$$

$$E_{\pm e_k \mp e_l} = (\mathcal{V}_{k,l}^{-} \pm i\mathcal{V}_{k,l}^{+}), \quad (3.10)$$

and

$$H_k = -i\mathcal{V}_{k,k}^{+}. \quad (3.11)$$

In Appendix A we give their commutation relations and their explicit form as linear differential operators on the manifold X^{4n-1} in the parametrization (2.10).

B. Properties of the Generators

As we are using the quaternionic unitary sphere (2.3) instead of the quaternionic projective space (2.1), we must be aware of the fact that the irreducible unitary representations of the group $Sp(1) \otimes Sp(n)$ are directly realized on the space $\mathcal{H}^L(X^{4n-1})$ spanned by functions (2.28). Nevertheless, the space $\mathcal{H}^L(X^{4n-1})$ should be reducible with respect to the action of the group $Sp(n)$. To show this, we use the formulas for the action of generators of algebras $\mathcal{R}_n$ and $\mathcal{R}_1'$ of $Sp(n)$ and $Sp(1)$, respectively, on the basis functions (2.28). They are collected in Appendix B, and one can easily see that the generators have the following properties.

(i) The generators H_p $(p = 1, \cdots, n)$ form the Cartan subalgebra of $\mathcal{R}_n$ and are diagonal in parametrization (2.10). They have the eigenvalues

$$M_p^{+} = m_p + \bar{m}_p. \quad (3.12)$$

(ii) The generators E_{2e_p} $(p = 1, \cdots, n)$ conserve all numbers L_i and l_i and also the value of

$$M_p^{-} = m_p - \bar{m}_p. \quad (3.13)$$

(iii) All remaining generators conserve the value of

$$M^{-} = \sum_{p=1}^{n} M_p^{-} = \sum_{p=1}^{n} (m_p - \bar{m}_p). \quad (3.14)$$

The last property is simply a consequence of the fact that the generator $\mathcal{W}_3$ of $Sp(1)$, which has the eigenvalue (3.14), commutes with the algebra $\mathcal{R}_n$ of $Sp(n)$. Therefore, the space $\mathcal{H}^L(X^{4n-1})$, spanned by functions (2.28), decomposes into subspaces $\mathcal{H}_{M^-}^L(X^{4n-1})$ of simultaneous eigenfunctions of the Laplace–Beltrami operator $\Delta(X^{4n-1})$ and of the generator $\mathcal{W}_3$ of $Sp(1)$.

Now, the value of M^- is restricted by the conditions (2.26) and (2.27), so that

$$|M^-| \leq L. \quad (3.15)$$

The structure and properties of subspaces $\mathcal{H}_{M^-}^L(X^{4n-1})$ strongly depend on the value of M^-. In the case when $|M^-| = L$, the subspaces $\mathcal{H}_{\pm L}^L(X^{4n-1})$ are irreducible under the action of the group $Sp(n)$, and therefore they can be considered as representation spaces for a unitary, irreducible representation of $Sp(n)$. Because these representations are characterized by a single number L, we call them "most degenerate representations." They will be treated in detail in this section. In the case when $|M^-| < L$, the situation is not so simple. The space $\mathcal{H}_{M^-}^L(X^{4n-1})$ is in this case reducible with respect to the action of the algebra of $Sp(n)$, and, to obtain its irreducible components, one needs further investigation. We have devoted Sec. 4 to these questions.

C. Unitarity and Irreducibility of the most Degenerate Representations of $Sp(n)$

The condition $M^- = +L$[14] reduces the two sets of equations (2.26) and (2.27) to

$$(m_p - \bar{m}_p) = l_p \quad (p = 1, \cdots, n) \quad (3.16)$$

and

$$L_{p-1} + l_p = L_p \quad (p = 2, \cdots, n). \quad (3.17)$$

[14] We are limiting ourselves only to the case $M^- = +L$ because the choice $M^- = -L$ leads only to the change of sign in the right-hand side of Eq. (3.16), which means changing the sign at any j's in lower indices of functions (3.18). The corresponding representations are equivalent.

These are strict conditions on the eigenvalues, and they select from the set of eigenfunctions (2.28) of the Laplace–Beltrami operator the subset of functions

$$
\begin{aligned}
& Y^{L,L_s;}_{M_s^+}(\Omega^{(n)}) \\
& \equiv Y^{L,L_{n-1},\cdots,L_2;2j_n,\cdots,2j_1}_{\frac{1}{2}M_n^++j_n,\cdots,\frac{1}{2}M_1^++j_1;\frac{1}{2}M_n^+-j_n,\cdots,\frac{1}{2}M_1^+-j_1}(\Omega)^{(n)} \\
& = (N_n)^{-\frac{1}{2}} \exp\{i[\tfrac{1}{2}M_1^+(\varphi_1+\psi_1)+j_1(\varphi_1-\psi_1)]\} \\
& \quad \times d^{j_1}_{\frac{1}{2}M_1^+,j_1}(2\vartheta_1) \\
& \quad \times \prod_{s=2}^{n} \exp\{i[\tfrac{1}{2}M_s^+(\varphi_s+\psi_s)+j_s(\varphi_s-\psi_s)]\} \\
& \quad \times d^{j_s}_{\frac{1}{2}M_s^+,j_s}(2\vartheta_s)\frac{(\sin\xi_s)^{3-2s}}{\cos\xi_s} d^{J_s}_{J_s-2J_{s-1},J_s}(2\xi_s). \qquad (3.18)
\end{aligned}
$$

Here we have introduced the notation

$$j_s = \tfrac{1}{2}(L_s - L_{s-1}) \qquad (3.19)$$

and

$$J_s = \tfrac{1}{2}(L_s + 2s - 2). \qquad (3.20)$$

In the special case $s = 1$ we have

$$j_1 = \tfrac{1}{2}L_1 \equiv \tfrac{1}{2}l_1 = J_1. \qquad (3.21)$$

We have also put $L \equiv L_n$.

In the considerations which follow a key role is played by the second-order invariant operator $\hat{I}_n^{(2)}$ which is proportional to the second-order Casimir operator of the group $Sp(n)$. We have found the following connection of this operator with the Laplace–Beltrami operator $\Delta(X^{4n-1})$ and the second-order invariant operator $\hat{I}_1^{(2)}$ of the $Sp(1)$ group which enters the direct product $Sp(1) \otimes Sp(n)$:

$$\hat{I}_n^{(2)} = \tfrac{1}{2}(\Delta + \hat{I}_1^{(2)}). \qquad (3.22)$$

Here we define

$$\hat{I}_n^{(2)} = \sum_{s=1}^{n} H_s^2 - \sum_{\alpha>0} [E_\alpha E_{-\alpha} + E_{-\alpha}E_\alpha] \qquad (3.23)$$

and

$$\hat{I}_1^2 = -\sum_{s=1}^{3} \mathfrak{W}_s^2, \qquad (3.24)$$

while Δ is defined by Eq. (2.12).

The functions (3.18) are simultaneous eigenfunctions of the operators Δ and $\hat{I}_1^{(2)}$, and therefore also of $\hat{I}_n^2$, the eigenvalues being $L(L + 4n - 2)$, $L(L + 2)$, and $L(L + 2n)$, respectively. They span a Hilbert space $\mathcal{H}_L^L(X^{4n-1})$ defined by the scalar product (2.31) with the left-invariant measure (2.33).

a. Unitarity

The space $\mathcal{H}_L^L(X^{4n-1})$ creates a representation space for the group $Sp(n)$ because, for any generator $Z_\alpha \in \mathfrak{R}_n$ and any $\eta \in \mathcal{H}_L^L(X^{4n-1})$, we have

$$[\Delta, Z_\alpha]\eta = 0, \quad [\hat{I}_n^{(2)}, Z_\alpha]\eta = 0$$

and

$$[\mathfrak{W}_3, Z_\alpha]\eta = 0. \qquad (3.25)$$

We shall denote representations of the group $Sp(n)$ related to this space by $D_L^L[Sp(n)]$ or simply D_L^L. They are realized by associating to any element $g \in Sp(n)$ an operator T_g in $\mathcal{H}_L^L(X^{4n-1})$ such that

$$(T_g\eta^{(L)})(\Omega) = \eta^{(L)}(g^{-1}\Omega) \qquad (3.26)$$

for any $\eta^{(L)}(\Omega) = \sum_{L_s}\sum_{M_s^+} c(L_s, M_s^+) Y^{L,L_s;}_{M_s^+}(\Omega)$ of $\mathcal{H}_L^L(X^{4n-1})$. Here Ω is a point of the manifold X^{4n-1}, and $g^{-1}\Omega$ is its left translation by the element g^{-1} of $Sp(n)$. Then unitarity of representations D_L^L follows immediately from the left invariance of the measure $d\mu(\Omega)$.

b. Irreducibility

From the explicit form (3.18) of the eigenfunctions $Y^{L,L_s;}_{M_s^+}$ we see that the structure of the Hilbert space $\mathcal{H}_L^L(X^{4n-1})$ is relatively simple. Namely, we can decompose $\mathcal{H}_L^L(X^{4n-1})$ into the direct sum of subspaces as follows:

$$\mathcal{H}_L^L(X^{4n-1}) = \bigoplus_{L_{n-1}} \bigoplus_{M_n^+} \mathcal{H}^{L;L_{n-1}}_{L;M_n^+}(X^{4n-1}), \qquad (3.27)$$

where the direct sum over L_{n-1} and M_n^+ is extended through

$$L_{n-1} = 0, 1, \cdots, L_n \qquad (3.28)$$

and

$$M_n^+ = -2j_n, \quad -2j_n + 2, \cdots, 2j_n, \qquad (3.29)$$

respectively. Each of the spaces $\mathcal{H}^{L;L_{n-1}}_{L;M_n^+}(X^{4n-1})$ forms a representation space for an irreducible unitary representation of the $Sp(n-1)$ subgroup of $Sp(n)$. We see that any representation of the maximal subgroup $Sp(n-1)$ occurs only once in the decomposition (3.23). This is illustrated diagrammatically in Fig. 1. To each point (L_{n-1}, l_n) in the diagram Fig. 1(a) there corresponds a diagram Fig. 1(b) which gives the possible values of m_n and $\bar{m}_n$ for a given l_n.

Now, to prove the irreducibility of representations D_L^L of $Sp(n)$, it is sufficient to prove that, starting from

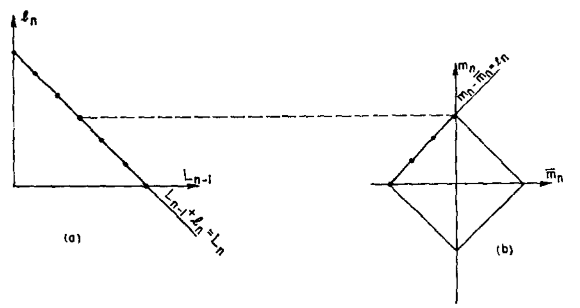

FIG. 1. The most degenerate representation of $Sp(n)$. (a) Decomposition with respect to the subgroup $Sp(1) \otimes Sp(n-1)$. (b) Possible values for m_n and $\bar{m}_n$ at a given l_n in the case when $M^- = L$.

any point in diagrams (a) and (b) of Fig. 1, we can, by successive application of the generators of the algebra $\mathfrak{R}_n$ of $Sp(n)$, reach any other point in these diagrams.

It is simple to prove this for $Sp(1)$ ($n = 1$) because the algebra of $Sp(1)$ is formed by the three generators $E_{\pm 2e_1}$ and H_1; the former act as step operators, while the latter is diagonal. The presence of the "stopping" factor $[(l \mp m \mp \bar{m})]^{\frac{1}{2}}$ in the formula (B3) of Appendix B for the action of the generators $E_{\pm 2e_1}$ assures us that, starting from some point of Fig. 1(b), we can reach any other point on the diagram and only these points. This means that the irreducibility of representation D_L^L ($L = l_1$) of $Sp(1)$ is proved. Similarly, one proves the irreducibility of the most degenerate representations for the $Sp(2)$ and $Sp(3)$ groups. Now let us assume that we have proved the irreducibility of D_L^L for the group $Sp(n-1)$. It means that the spaces $\mathcal{K}_{L;M_n^+}^{L;L_{n-1}}(X^{4n-1})$ are irreducible with respect to the action of the generators of the algebra $\mathfrak{R}_{n-1}$ of $Sp(n-1)$.

Now, to prove the irreducibility of $\mathcal{K}_L^L(X^{4n-1})$ space with respect to the actions of the algebra $\mathfrak{R}_n$ of $Sp(n)$, it is sufficient to consider only the action of generators $E_{\pm 2e_n}$, $E_{\pm e_n \pm e_{n-1}}$, and $E_{\pm e_n e \mp_{n-1}}$ (see also Appendix B). The corresponding formulas are formulas (B11)–(B14) and (B3) of Appendix B. We see that the considered generators conserve the conditions (3.16) and (3.17). Moreover, the stopping factors are always combined in such a way that relations

$$l_n \geq 0; \quad L_{n-1} \geq 0$$

and

$$|m_n + \bar{m}_n| \leq l_n; \quad |m_{n-1} + \bar{m}_{n-1}| \leq l_{n-1}$$

are fulfilled. This procedure enables us to prove the irreducibility of representations D_L^L of $Sp(n)$ by full induction.

D. Properties of the Most Degenerate Representations of $Sp(n)$

The basis functions (3.18) of the Hilbert space $\mathcal{K}_L^L(X^{4n-1})$ are characterized by n numbers L_s and n numbers M_s^+ ($s = 1, \cdots, n$). The former are related to the eigenvalues of the set of the second-order invariant operators $I_s^{(2)}$ of the chain of subgroups

$$Sp(n) \supset Sp(n-1) \supset \cdots \supset Sp(1). \tag{3.30}$$

These eigenvalues are

$$\lambda_s = L_s(L_s + 2s). \tag{3.31}$$

Numbers M_s^+ are then eigenvalues of the generators H_s of the Cartan subgroup of $Sp(n)$.

Therefore, the set of commuting operators is explicitly formed by $2n$ operators:

$$\{I_n^{(2)}, H_n, I_{n-1}^{(2)}, H_{n-1}, \cdots, I_1^{(2)}, H_1\}. \tag{3.32}$$

Their number is reasonably small compared to the corresponding number in the case of a nondegenerate series of representations of $Sp(n)$, in which case it is $\frac{1}{2}n(n+5) - 1$. This may be of particular interest from the point of view of physical applications because we would usually like to have the smallest possible number of invariants for characterization of a given physical state. The fact that all Cartan subgroup generators are diagonal, which is due to the parametrization we have employed, is also convenient because it makes it possible to relate each of these generators to an additive conservation law.

Another property which is a direct consequence of the choice of the parametrization of the domain X^{4n-1} is the pattern of the decomposition of the Hilbert space $\mathcal{K}_L^L(X^{4n-1})$ into the subspaces $\mathcal{K}_{L,M_n^+}^{L,L_{n-1}}(X^{4n-1})$ on which the subgroup $Sp(n-1)$ acts irreducibly. One can easily find a parametrization in which the representation D_L^L of $Sp(n)$ decomposes similarly with respect to a subgroup $Sp(n-k)$ with arbitrary k ($n > k \geq 1$).

In Appendix C we give the detailed calculation of the highest weight of the representation of $Sp(n)$. From it, it follows that the representation D_L^L corresponds to the representation $D(L, 0, \cdots, 0)$ in the notation used in Ref. 15, or it can be represented by a one-row Young tableaux (see Ref. 16) and therefore may be interpreted as a fully symmetrical representation of $Sp(n)$.[17]

4. LESS DEGENERATE REPRESENTATIONS OF $Sp(n)$ AND THE REPRESENTATIONS OF THE GROUP $Sp(1) \otimes Sp(n)$

A. The Less Degenerate Representations of $Sp(n)$

The condition

$$|M^-| = \text{const} < L$$

imposes a restriction on the eigenvalues, which can be conventionally written as

$$|M^-| \leq l_n + L_{n-1}. \tag{4.1}$$

But from (2.27) it follows that

$$|M^-| \leq L - 2k, \tag{4.2}$$

[15] M. Konuma, K. Shima, and M. Wada, Progr. Theoret. Phys. (Kyoto) Suppl., No. 28 (1963).

[16] M. Hamermesh, *Group Theory and Its Applications* (Addison-Wesley Publishing Company, Inc., London, 1962).

[17] The explicit examples of lowest representations of $Sp(2)$ and $Sp(3)$ have been published in Appendix IV to the preprint [IC/66/77, ICTP, Trieste (1966)] of this paper.

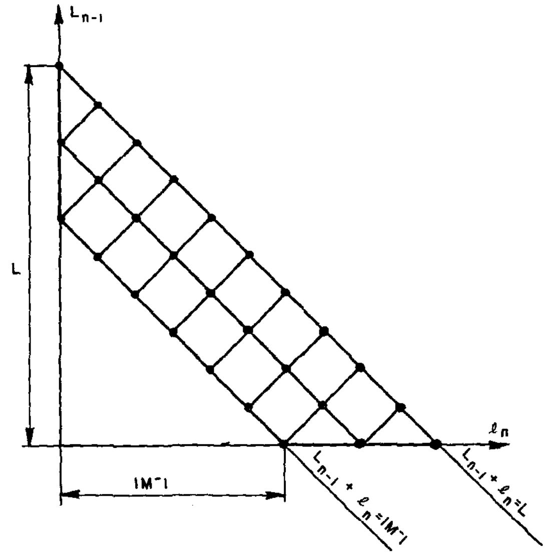

FIG. 2. Diagram of the structure of the space $\mathcal{H}^L_{M^-}$.

where now k may be one of the numbers

$$k = 0, 1, \cdots, \left[\frac{L - |M^-|}{2}\right]. \tag{4.3}$$

We can again represent the possible values of L_{n-1} and l_n by points in the net of Fig. 2. From the formula (B13) of Appendix B we see that the operator $\hat{I}_1^{(2)}$ and (therefore also) the second-order invariant operator $\hat{I}_n^{(2)}$ of $Sp(n)$ are not diagonal on the space $\mathcal{H}^L_{M^-}(X^{4n-1})$. Generally, the operators $\hat{I}_1^{(2)}$ and $\hat{I}_n^{(2)}$, when acting on functions (2.28), conserve, besides all the numbers L_s and l_s ($s = 1, \cdots, n$), also the numbers M_s^+ ($s = 1, \cdots, n$) and M^-. Generally they change the values of M_s^- ($s = 1, \cdots, n$). The last numbers are eigenvalues of operators $\tilde{H}_s$ which do not belong to the set of commuting operators of the algebra $\mathcal{R}_n$ of $Sp(n)$. Therefore, it is necessary to diagonalize the operators $\hat{I}_1^{(2)}$ and $\hat{I}_n^{(2)}$ on the subspaces $\mathcal{H}^{L;L_s}_{M_s^+,M^-}(X^{4n-1})$. This leads to the usual procedure of diagonalization of matrices which represent the generators on a given representation space.

In order to see the structure of the space $\mathcal{H}^L_{M^-}(X^{4n-1})$, let us first investigate the points for which $L_{n-1} = 0$. Then we have all numbers equal to zero except l_n and M_n^+. The value of M_n^- is fixed by

$$M_n^- = M^-. \tag{4.4}$$

Therefore the condition (4.4) defines the function $Y^{L,0,\cdots,0;l_n,0,\cdots,0}_{M_n^+,0,\cdots,0;M_n^-,0,\cdots,0}(\Omega^{(n)})$. The eigenvalue of $\hat{I}_1^{(2)}$ will then be

$$\hat{I}_1^{(2)} = l_n(l_n + 2). \tag{4.5}$$

This means that functions with $L_{n-1} = 0$ and different values of l_n in $\mathcal{H}^L_{M^-}(X^{4n-1})$ belong to different irreducible representations of $Sp(n)$. We denote the eigenvalue of $\hat{I}_1^{(2)}$ by $\tilde{L}(\tilde{L} + 2)$, and then from (4.5) and (4.1) we get

$$\tilde{L} = |M^-|, |M^-| + 2, \cdots, L. \tag{4.6}$$

Now taking into account the generators of the algebra $\mathcal{R}_n$, we see that to the same representation as the point $(\tilde{L}, 0)$ in diagram (L_{n-1}, l_n) must also belong the points $(\tilde{L} \pm 1, 1)$. Proceeding in this way, we get the following decomposition of the space $\mathcal{H}^L_{M^-}(X^{4n-1})$ into the subspaces which are irreducible under the action of $Sp(n)$:

$$\mathcal{H}^L_{M^-}(X^{4n-1}) = \bigoplus_{\tilde{L}=|M^-|}^{L} \mathcal{H}^L_{\tilde{L},M^-}(X^{4n-1}). \tag{4.7}$$

This is illustrated diagrammatically in Fig. 3. Of course, the set of functions which span the space $\mathcal{H}^L_{\tilde{L},M^-}$ is obtained by the diagonalization of the operator $\hat{I}_1^{(2)}$ in all subspaces $\mathcal{H}^{L,L_s,l_s}_{M_s^+,M^-}$. Therefore, the function ${}^{\tilde{L},\tilde{L}_s}_{M^-}Y^{L,L_s;l_s}_{M_s^+,}(\Omega)$ is the simultaneous eigenfunction of the set of $4n - 1$ operators

$$\{\mathcal{W}_3; \Delta(X^{4n-1}), \hat{I}_n^{(2)}, \hat{K}_n, H_n; \Delta(X^{4(n-1)-1}), \\ \hat{I}_{n-1}^{(2)}, \hat{K}_{n-1}, H_{n-1}, \cdots, \hat{I}_1^{(2)}, H_1\} \tag{4.8}$$

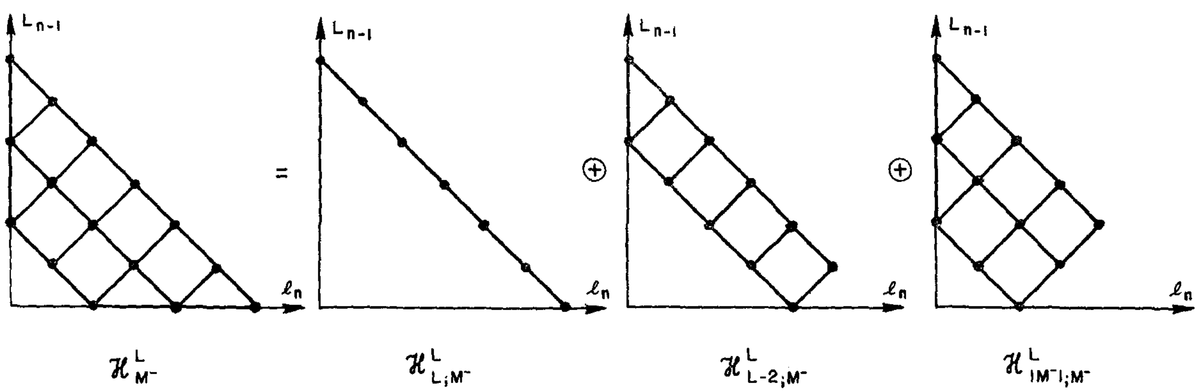

FIG. 3. Decomposition of the space $\mathcal{H}^L_{M^-}$ into the subspaces irreducible under the action of the algebra $\mathcal{R}_n$ of $Sp(n)$.

and is generally expressed as a certain linear combination of functions which span the space $\mathcal{H}^{L,L_s;l_s}_{M_s^+,M^-}$. It will be given explicitly in the forthcoming paper on noncompact groups.

On the spaces $\mathcal{H}^{L}_{\tilde{L},M^-}(X^{4n-1})$ one can introduce the irreducible unitary representations of the $Sp(n)$. Unitarity of these representations can be proved in the same way as it was in the case of the most degenerate representations.

The highest weight of a representation $D^L_{\tilde{L}}$ realized on the space $\mathcal{H}^{L}_{\tilde{L},M^-}(X^{4n-1})$ is calculated in Appendix C. Then the representation $D^L_{\tilde{L}}$ can be denoted, according to the notation of Konuma, Shima, and Wada,[15] by

$$D^L_{\tilde{L}} = D\left(\tilde{L}, \frac{L-\tilde{L}}{2}, 0, \cdots, 0\right), \tag{4.9}$$

which corresponds to the Young tableaux with two rows defined by the symmetry scheme[17]

$$\left[\frac{L+\tilde{L}}{2}, \frac{L-\tilde{L}}{2}, 0, \cdots, 0\right]. \tag{4.10}$$

The representations realized on spaces $\mathcal{H}^{L}_{\tilde{L},M^-_{(1)}}$ and $\mathcal{H}^{L}_{\tilde{L},M^-_{(2)}}$, where

$$M^-_{(1)} \neq M^-_{(2)},$$

are equivalent because they have the same highest weight.

B. Representations of $Sp(1) \otimes Sp(n)$

Finally, we should remark that the full set of functions (2.28) spans a Hilbert space $\mathcal{H}^L(X^{4n-1})$ which acts as a carrier space for irreducible unitary representations of the group $Sp(1) \otimes Sp(n)$. The corresponding decomposition of the Hilbert space $\mathcal{H}^L(X^{4n-1})$ into subspaces is then given by

$$\mathcal{H}^L(X^{4n-1}) = \bigoplus_{L_{n-1}} \bigoplus_{l_n} \bigoplus_{m_n} \bigoplus_{\tilde{m}_n} \mathcal{H}^{L;L_{n-1};l_n}_{m_n,\tilde{m}_n}(X^{4n-1}), \tag{4.11}$$

where the summation is restricted by conditions (2.26) and (2.27). There is no need to repeat the same kind of considerations as above to prove the irreducibility and unitarity of the representations D^L of $Sp(1) \otimes Sp(n)$ on this space. Since $Sp(1) \otimes Sp(n)$ is a direct product, and we have treated $Sp(n)$ in the previous sections, it is only necessary to use generators from both $\mathcal{R}_1'$ and $\mathcal{R}_n$ algebras of $Sp(1)$ and $Sp(n)$, respectively, to prove the irreducibility of representation D^L. Representations D^L of $Sp(1) \otimes Sp(n)$ decompose into the representations of the subgroups $Sp(1)$ and $Sp(n)$ according to the formula

$$D^L[Sp(1) \otimes Sp(n)] = \bigoplus_{\tilde{L}} \{D^{\tilde{L}}[Sp(1)] \otimes D^L_{\tilde{L}}[Sp(n)]\}, \tag{4.12}$$

as may be easily verified.

5. CONCLUSION

The main results can be summarized in the following way.

(i) A set of harmonic functions which span the representation space for the most degenerate unitary irreducible representations of the compact symplectic group $Sp(n)$ has been found. These representations are characterized by a single number which is related to the eigenvalue of the second-order Casimir operator of $Sp(n)$. They can be also described by a one-row Young tableaux.

(ii) Besides this, the series of unitary irreducible representations of $Sp(n)$ characterized by two independent numbers L and $\tilde{L}$ has been obtained. These numbers are related to the eigenvalues of the Laplace–Beltrami operator $\Delta(X^{4n-1})$ on the quaternionic unitary sphere X^{4n-1} and of the second-order Casimir operator of $Sp(n)$, respectively. To these representations correspond the Young tableaux with two rows.

(iii) The number of operators in the maximal set of commuting operators, the eigenvalues of which characterize the basis functions of the representation space, in the case of the most degenerate representations is

$$N_1 = 2n, \tag{5.1}$$

and in the case of the less degenerate representations it is

$$N_2 = 4n - 1. \tag{5.2}$$

These numbers are sufficiently small when compared to the corresponding number in the case of nondegenerate representations of $Sp(n)$:

$$N_n = \tfrac{1}{2}n(n+5) - 1. \tag{5.3}$$

This underlines the importance of the degenerate representations for physical applications.

(iv) In the parametrization introduced on the quaternionic unitary sphere X^{4n-1}, the generators H_s which form the Cartan subalgebra are all diagonal, which, again, is useful for applications.

(v) Finally, the patterns of the decomposition of the given irreducible representation of $Sp(n)$, with respect to the maximal compact subgroup $Sp(1) \otimes Sp(n-1)$ of it, occur because of the parametrization.

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APPENDIX A: ALGEBRA OF $Sp(n)$ AND OF $Sp'(1)$

In order to obtain the algebras of $Sp(n)$ and $Sp'(1)$ groups, let us consider an infinitesimal symplectic transformation in the unitary n-dimensional quaternionic space $\mathfrak{Q}^{(n)}$:

$$A\mathbf{q} \to \mathbf{q} \quad \text{for} \quad \mathbf{q}, \mathbf{q}' \in \mathfrak{Q}^{(n)}. \tag{A1}$$

The necessary and sufficient condition for the $n \times n$ matrix of quaternionic elements A to be symplectic is

$$AA^+ = A^+A = I, \tag{A2}$$

where

$$(A^+)_{st} = \bar{A}_{ts}. \tag{A3}$$

We point out that the "plus sign" here represents the quaternionic conjugate defined in Sec. 3 and the transposition of matrix A. Representing any element of A in the form

$$A_{st} = a_{st} + ib_{st} + jc_{st} + kd_{st} \tag{A4}$$

and considering the real parameters a_{st}, b_{st}, c_{st} and d_{st} $(s, t = 1, \cdots, n)$ as infinitesimally small quantities, we obtain, imposing condition (A2),

$$a_{ss} = 1 \quad (s = 1, \cdots, n), \tag{A5}$$

$$\left.\begin{aligned} a_{st} &= -a_{ts} \quad (s \neq t), \\ b_{st} &= b_{ts}, \\ c_{st} &= c_{ts}, \\ d_{st} &= d_{ts}, \end{aligned}\right\} \quad (s, t = 1, \cdots, n), \tag{A6}$$

or

$$A_{st} = -\bar{A}_{ts}, \quad \text{for} \quad s \neq t.$$

Let us now define four formal, linearly independent, quaternionic quantities:

$$\begin{aligned} q_{s+} &= a_s + ib_s + jc_s + kd_s = z_s + z_{-s}j, \\ q_{s-} &= a_s + ib_s - jc_s - kd_s = z_s - z_{-s}j, \\ q^*_{s+} &= a_s - ib_s + jc_s - kd_s = z^*_s + z^*_{-s}j, \\ q^*_{s-} &= a_s - ib_s - jc_s + kd_s = z^*_s - z^*_{-s}j. \end{aligned} \tag{A7}$$

We define the representation of the group $Sp(n)$ as a transformation in the space of functions of these variables determined by

$$\begin{aligned} (T_A f)(q_{1+}, \cdots, q^*_{1-}; \cdots; q_{n+}, \cdots, q^*_{n-}) \\ = f[(A^{-1}q_1)_+, \cdots, (A^{-1}q_1)^*_-; \cdots; \\ (A^{-1}q_n)_+, \cdots, (A^{-1}q_n)^*_-] \end{aligned} \tag{A8}$$

for any $A \in Sp(n)$. Defining the formal derivatives by

$$\begin{aligned} \partial_{s+} &= \frac{\partial}{\partial q_{s+}} = \frac{1}{2}\left(\frac{\partial}{\partial z_s} - j\frac{\partial}{\partial z_{-s}}\right), \\ \partial_{s-} &\equiv \frac{\partial}{\partial q_{s-}} = \frac{1}{2}\left(\frac{\partial}{\partial z_s} + j\frac{\partial}{\partial z_{-s}}\right), \\ \partial^*_{s+} &\equiv \frac{\partial}{\partial q^*_{s+}} = \frac{1}{2}\left(\frac{\partial}{\partial z^*_s} - j\frac{\partial}{\partial z^*_{-s}}\right), \\ \partial^*_{s-} &\equiv \frac{\partial}{\partial q^*_{s-}} = \frac{1}{2}\left(\frac{\partial}{\partial z^*_s} + j\frac{\partial}{\partial z^*_{-s}}\right), \end{aligned} \tag{A9}$$

we can expand the right-hand side of Eq. (A8) into a Taylor series. Then the generators of the $Sp(n)$ group are obtained if we consider the one-parameter subgroups of $Sp(n)$. This procedure leads to the following set of generators:

$$\begin{aligned} L_{a_{st}} &= [q_t\partial_s(++++) - q_s\partial_t(++++)] \equiv 2\mathcal{V}^-_{ts}, \\ L_{b_{ss}} &= i[q_s\partial_s(++--)] \equiv \mathcal{V}^+_{ss}, \\ L_{b_{st}} &= i[q_t\partial_s(++--) + q_s\partial_t(++--)] \equiv 2\mathcal{V}^+_{ts}, \\ L_{c_{ss}} &= j[q_s\partial_s(+-+-)] \equiv \mathcal{U}^+_{ss}, \\ L_{c_{st}} &= j[q_t\partial_s(+-+-) + q_s\partial_t(+-+-)] \equiv \mathcal{U}^+_{ts}, \\ L_{d_{ss}} &= k[q_s\partial_s(+--+)] \equiv \mathcal{U}^-_{ss}, \\ L_{d_{st}} &= k[q_t\partial_s(+--+) + q_s\partial_t(+--+)] \equiv \mathcal{U}^-_{ts}. \end{aligned} \tag{A10}$$

Here we have introduced the notation that, for example, $q_s\partial_t(+--+)$ represents the expression

$$q_{s+}\partial_{t+} - q_{s-}\partial_{t-} - q^*_{s+}\partial^*_{t+} + q^*_{s-}\partial^*_{t-}.$$

The commutation relations (3.3)–(3.5) of the generators $\mathcal{U}^\pm_{st}$ and $\mathcal{V}^\pm_{st}$, which are given in Sec. 3, are easily verified either by using the definition of quaternionic derivatives (A9) or by expressing generators $\mathcal{U}^\pm_{st}$ and $\mathcal{V}^\pm_{st}$ in variables z_s, z_{-s}, z^*_s, and z^*_{-s} as follows:

$$\begin{aligned} \mathcal{U}^+_{st} &= \frac{1}{2}\left[z_s\frac{\partial}{\partial z_{-t}} + z_t\frac{\partial}{\partial z_{-s}} - z_{-s}\frac{\partial}{\partial z_t} \right. \\ &\qquad \left. - z_{-t}\frac{\partial}{\partial z_s} + \text{complex conjugate}\right], \\ \mathcal{U}^-_{st} &= \frac{1}{2i}\left[z_s\frac{\partial}{\partial z_{-t}} + z_t\frac{\partial}{\partial z_{-s}} + z_{-s}\frac{\partial}{\partial z_t} + z_{-t}\frac{\partial}{\partial z_s} - \text{c.c.}\right], \\ \mathcal{V}^+_{st} &= \frac{1}{2i}\left[z_s\frac{\partial}{\partial z_t} + z_t\frac{\partial}{\partial z_s} - z_{-s}\frac{\partial}{\partial z_{-t}} - z_{-t}\frac{\partial}{\partial z_{-s}} - \text{c.c.}\right], \\ \mathcal{V}^-_{st} &= \frac{1}{2}\left[z_s\frac{\partial}{\partial z_t} - z_t\frac{\partial}{\partial z_s} + z_{-s}\frac{\partial}{\partial z_{-t}} - z_{-t}\frac{\partial}{\partial z_{-s}} + \text{c.c.}\right]. \end{aligned} \tag{A11}$$

Now we may express these generators through those of $\mathcal{U}(2n)$ given in Ref. 4 as

$$L^{+}_{s,t} = \left[z_s \frac{\partial}{\partial z_t} - z_t \frac{\partial}{\partial z_s} + \text{c.c.}\right],$$

$$L^{-}_{s,t} = -i\left[z_s \frac{\partial}{\partial z_t} + z_t \frac{\partial}{\partial z_s} - \text{c.c.}\right]. \quad \text{(A12)}$$

We see that

$$\mathcal{U}^{\pm}_{s,t} = \tfrac{1}{2}[L^{\pm}_{s,-t} + L^{\pm}_{t,-s}], \quad \text{(A13)}$$

$$\mathcal{V}^{\pm}_{s,t} = \tfrac{1}{2}[L^{\mp}_{s,t} \mp L^{\mp}_{-s,-t}]. \quad \text{(A14)}$$

Because the algebra $\mathcal{R}_n$ of the $Sp(n)$ group is the compact real form of the algebra of the complex group C_n, we can easily find the Weyl's standard basis H_k, $E_{\pm 2e_k}$, $E_{\pm e_k \pm e_l}$, $E_{\pm e_k \mp e_l}$ of C_n by using the generators (A10). This relation is given by formulas (3.8)–(3.11) of Sec. 3.

The commutation relations of these generators are

$$\begin{aligned}
[E_\alpha, E_{-\alpha}] &= \sum_{k=1}^{n} \alpha_k H_k, \\
[E_\alpha, E_\beta] &= N_{\alpha,\beta} E_{\alpha+\beta}, \\
[H_k, E_\alpha] &= -\alpha_k E_\alpha, \\
[H_k, H_l] &= 0,
\end{aligned} \quad \text{(A15)}$$

where α and β are roots and where $N_{\alpha,\beta} = N_{-\alpha,-\beta} \neq 0$ only if $\alpha + \beta$ is also a root. We obtain a Dynkin diagram (see Fig. 4) of the group C_n if we put

$$\begin{aligned}
\boldsymbol{\alpha}^{(1)} &= \mathbf{e}_n - \mathbf{e}_{n-1}, \\
&\cdot \\
&\cdot \\
&\cdot \\
\boldsymbol{\alpha}^{(n-1)} &= \mathbf{e}_2 - \mathbf{e}_1, \\
\boldsymbol{\alpha}^{(n)} &= 2\mathbf{e}_1,
\end{aligned} \quad \text{(A16)}$$

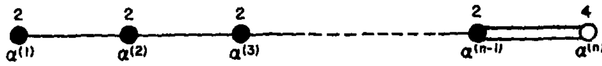

FIG. 4. Dynkin diagram of the group C_n.

where $\mathbf{e}_1, \cdots, \mathbf{e}_n$ are unit vectors which form an orthonormal basis in the root space.

Generators $E_{\pm\alpha}$ play the role of raising and lowering operators when acting on the basis functions (2.28) of the space $\mathcal{H}^L(X^{4n-1})$, while the generators H_k form a Cartan subalgebra. In the following we present explicit expressions for these generators as linear differential operators in the parametrization of the manifold X^{4n-1} which we introduced in Sec. 2. Similarly, the three generators of C_1 which correspond to the $Sp'(1)$ group are defined by

$$E_{\pm 2e_0} = 2^{-\frac{1}{2}} \sum_{k=1}^{n} (\tilde{\mathcal{U}}^{+}_{k,k} \pm i\tilde{\mathcal{U}}^{-}_{k,k}) = 2^{-\frac{1}{2}} \sum_{k=1}^{n} E^{k}_{\pm 2e_0},$$

$$H_0 = -i\mathcal{W}_3 = -i\sum_{k=1}^{n} \tilde{\mathcal{V}}^{+}_{k,k}, \quad \text{(A17)}$$

where

$$\begin{aligned}
\tilde{\mathcal{U}}^{+}_{k,k} &= \left[z_k \frac{\partial}{\partial z^*_{-k}} - z^*_{-k} \frac{\partial}{\partial z_k} + \text{c.c.}\right], \\
\tilde{\mathcal{U}}^{-}_{k,k} &= -i\left[z_k \frac{\partial}{\partial z^*_{-k}} + z^*_{-k} \frac{\partial}{\partial z_k} - \text{c.c.}\right], \\
\tilde{\mathcal{V}}^{+}_{k,k} &= -i\left[z_k \frac{\partial}{\partial z_k} - z^*_{-k} \frac{\partial}{\partial z^*_{-k}} - \text{c.c.}\right].
\end{aligned} \quad \text{(A18)}$$

In the parametrization of X^{4n-1} we now have

$$H_k = i\left(\frac{\partial}{\partial \varphi_k} - \frac{\partial}{\partial \psi_k}\right), \quad \text{(A19)}$$

$$E_{\pm 2e_k} = 2^{-\frac{1}{2}} \exp\{\pm i(\varphi_k - \psi_k)\} \times \left[\mp i \tan\vartheta_k \frac{\partial}{\partial \varphi_k} \mp i \cot\vartheta_k \frac{\partial}{\partial \psi_k} + \frac{\partial}{\partial \vartheta_k}\right]. \quad \text{(A20)}$$

For $k < l$ we have

$$\begin{aligned}
E_{\pm e_k \pm e_l} = \frac{1}{2}\Bigg\{ &\frac{f_l}{f_k}\Bigg[\mp i e^{\pm i(\varphi_k - \psi_l)} \frac{\sin\vartheta_l}{\cos\vartheta_k} \frac{\partial}{\partial \varphi_k} \mp i e^{\pm i(\varphi_l - \psi_k)} \frac{\cos\vartheta_l}{\sin\vartheta_k} \frac{\partial}{\partial \psi_k} \\
&+ (e^{\pm i(\varphi_l - \psi_k)} \cos\vartheta_k \cos\vartheta_l + e^{\pm i(\varphi_k - \psi_l)} \sin\vartheta_k \sin\vartheta_l) \frac{\partial}{\partial \vartheta_k}\Bigg] \\
&+ \frac{f_k}{f_l}\Bigg[\mp i e^{\pm i(\varphi_l - \psi_k)} \frac{\sin\vartheta_k}{\cos\vartheta_l} \frac{\partial}{\partial \varphi_l} \mp i e^{\pm i(\varphi_k - \psi_l)} \frac{\cos\vartheta_k}{\sin\vartheta_l} \frac{\partial}{\partial \psi_l} \\
&+ (e^{\pm i(\varphi_k - \psi_l)} \cos\vartheta_k \cos\vartheta_l + e^{\pm i(\varphi_l - \psi_k)} \sin\vartheta_k \sin\vartheta_l) \frac{\partial}{\partial \vartheta_l}\Bigg] \\
&- (e^{\pm i(\varphi_l - \psi_k)} \sin\vartheta_k \cos\vartheta_l - e^{\pm i(\varphi_k - \psi_l)} \cos\vartheta_k \sin\vartheta_l) G_{l,k}\Bigg\}
\end{aligned} \quad \text{(A21)}$$

and

$$E_{\mp e_k \pm e_l} = \frac{1}{2}\Bigg\{\frac{f_l}{f_k}\Bigg[\pm i e^{\pm i(\varphi_l-\varphi_k)}\frac{\cos\vartheta_l}{\cos\vartheta_k}\frac{\partial}{\partial\varphi_k} \mp i e^{\mp i(\psi_l-\psi_k)}\frac{\sin\vartheta_l}{\sin\vartheta_k}\frac{\partial}{\partial\psi_k}$$
$$+ (e^{\pm i(\varphi_l-\varphi_k)}\cos\vartheta_l\sin\vartheta_k - e^{\mp i(\psi_l-\psi_k)}\sin\vartheta_l\cos\vartheta_k)\frac{\partial}{\partial\vartheta_k}\Bigg]$$
$$+\frac{f_k}{f_l}\Bigg[\pm i e^{\pm i(\varphi_l-\varphi_k)}\frac{\cos\vartheta_k}{\cos\vartheta_l}\frac{\partial}{\partial\varphi_l} \mp i e^{\mp i(\psi_l-\psi_k)}\frac{\sin\vartheta_k}{\sin\vartheta_l}\frac{\partial}{\partial\psi_l}$$
$$- (e^{\pm i(\varphi_l-\varphi_k)}\sin\vartheta_l\cos\vartheta_k - e^{\mp i(\psi_l-\psi_k)}\cos\vartheta_l\sin\vartheta_k)\frac{\partial}{\partial\vartheta_l}\Bigg]$$
$$+ (e^{\pm i(\varphi_l-\varphi_k)}\cos\vartheta_l\cos\vartheta_k + e^{\mp i(\psi_l-\psi_k)}\sin\vartheta_l\sin\vartheta_k)G_{l,k}\Bigg\}. \tag{A22}$$

Here

$$\frac{f_l}{f_k} = \frac{\cos\xi_l}{\cos\xi_k\sin\xi_{k+1}\cdots\sin\xi_l} \tag{A23}$$

and

$$G_{l,k} = \frac{\cos\xi_l\sin\xi_k}{\sin\xi_{k+1}\cdots\sin\xi_l}\frac{\partial}{\partial\xi_k} - \sum_{r=k+1}^{l-1}\frac{\cos\xi_l\cos\xi_k\sin\xi_{k+1}\cdots\sin\xi_{r-1}\cos\xi_r}{\sin\xi_{r+1}\cdots\sin\xi_l}\frac{\partial}{\partial\xi_r} - \cos\xi_k\sin\xi_{k+1}\cdots\sin\xi_{l-1}\frac{\partial}{\partial\xi_l}. \tag{A24}$$

The algebra of $Sp'(1)$ in $Sp'(1)\otimes Sp(n)$ is formed by the three generators

$$E_{\pm 2e_0} = 2^{-\frac{1}{2}}\sum_{k=1}^{n} e^{\pm i(\varphi_k+\psi_k)} \times\left[\pm i\tan\vartheta_k\frac{\partial}{\partial\varphi_k} \mp i\cot\vartheta_k\frac{\partial}{\partial\psi_k} + \frac{\partial}{\partial\vartheta_k}\right] \tag{A25}$$

and

$$H_0 = +i\sum_{k=1}^{n}\left(\frac{\partial}{\partial\varphi_k} + \frac{\partial}{\partial\psi_k}\right). \tag{A26}$$

It is rather easy to show that these generators commute with the algebra of $Sp(n)$.

APPENDIX B: ACTIONS OF GENERATORS ON BASIC FUNCTIONS

To prove irreducibility of the representations $D_L^L(X^{4n-1})$ and $D_{\bar L}^L(X^{4n-1})$ we need explicit formulas for the action of different generators of the algebras $\mathfrak{R}_1'$ and $\mathfrak{R}_n$ of $Sp'(1)$ and $Sp(n)$, respectively. However, we do not need to know this action for all generators of $Sp(n)$ because, if we know, for instance, the action of $E_{\pm e_p \pm e_{p-1}}$, $E_{\pm e_p \mp e_{p-1}}$, and E_{2e_p} for any $p = 2, \cdots, n$, we can use the commutation relations (A15) to get, for example, $E_{\pm e_p \pm e_{p-k}}$ for any $k < p$. For this reason in this appendix we give the formulas for the action of the generators $E_{\pm e_p \pm e_{p-1}}$, $E_{\pm e_p \mp e_{p-1}}$, and $E_{\pm 2e_p}$ on the basis functions (2.28).

As may easily be seen from the explicit expressions of the generators in Appendix A, the generators $E_{\pm 2e_p}$ do not change values of either L_k ($k = 2, \cdots, n$) or l_k ($k = 1, \cdots, n$), while $E_{\pm e_p \pm e_{p-1}}$ and $E_{\pm e_p \mp e_{p-1}}$ in these sets of eigenvalues change only L_{p-1}, l_p, and l_{p-1}. All the generators, except those of H_p which form the Cartan subalgebra, change the values of m_p, $\bar m_p$, m_{p-1}, and $\bar m_{p-1}$. Therefore, in what follows, we label the eigenfunctions only by the eigenvalues $L_p \equiv L$, $L_{p-1} \equiv L'$, $l_p \equiv l$, $l_{p-1} \equiv l'$, $m_p \equiv m$, $m_{p-1} \equiv m'$, $\bar m_p \equiv \bar m$, and $\overline{m_{p-1}} \equiv \bar m'$. Thus we have

$$H_p Y^{L,L';l,l';}_{m,m';\bar m,\bar m'} = (m - \bar m) Y^{L,L';l,l';}_{m,m';\bar m,\bar m'}, \tag{B1}$$

$$\tilde H_p Y^{L,L';l,l';}_{m,m';\bar m,\bar m'} = (m + \bar m) Y^{L,L';l,l'}_{m,m';\bar m,\bar m'}, \tag{B2}$$

$$E_{\pm 2e_p} Y^{L,L';l,l';}_{m,m';\bar m,\bar m'} = 2^{-\frac{1}{2}}[(l \mp m \pm \bar m)(l \pm m \mp \bar m + 2)]^{\frac{1}{2}} Y^{L,L';l,l';}_{m\pm1,m';\bar m\mp1,\bar m'}, \tag{B3}$$

$$\tilde E^{p}_{\pm 2e_0} Y^{L,L';l,l';}_{m,m';\bar m,\bar m',} = 2^{-\frac{1}{2}}[(l \mp m \mp \bar m)(l \pm m \pm \bar m + 2)] Y^{L,L';l,l';}_{m\pm1,m';\bar m\pm1,\bar m'}, \tag{B4}$$

$$\begin{aligned} E_{\pm e_p \pm e_{p-1}} Y^{L,L'l,l,';}_{m,m',\bar m,\bar m'} = {} & 16[(l+1)(l'+1)(L'+2p-3)]^{-\frac{1}{2}} \\ & \times \sum_{\delta L'=\pm1}\sum_{\delta l=\pm1}\sum_{\delta l'=\pm1} \operatorname{sign}(\delta L') a_{\delta L',\delta l} b_{\delta L',\delta l'} (\tilde c^{\mp}_{\delta l} d^{\pm}_{\delta l'} Y^{L,L'+\delta L';l+\delta l,l'+\delta l'}_{m,m'\pm1;\bar m\mp1,\bar m'} \\ & + c^{\pm}_{\delta l} \tilde d^{\mp}_{\delta l'} Y^{L,L'+\delta L',l+\delta l,l'+\delta l'}_{m\pm1,m';\bar m,\bar m'\mp1}), \end{aligned} \tag{B5}$$

$$\begin{aligned} E_{\mp e_p \pm e_{p-1}} Y^{L,L';l,l'}_{m,m';\bar m,\bar m'} = {} & 16[(l+1)(l'+1)(L'+2p-3)]^{-\frac{1}{2}} \\ & \times \sum_{\delta L'=\pm1}\sum_{\delta l=\pm1}\sum_{\delta l'=\pm1} \operatorname{sign}(\delta L') a_{\delta L',\delta l} b_{\delta L',\delta l'} (c^{\pm}_{\delta l} d^{\mp}_{\delta l'} Y^{L,L'+\delta L';l+\delta l,l'+\delta l'}_{m\pm1,m'\mp1;\bar m,\bar m'} - \tilde c^{\mp}_{\delta l} \tilde d^{\pm}_{\delta l'} Y^{L,L'+\delta L';l+\delta l,l'+\delta l'}_{m,m';\bar m\mp1,\bar m'\pm1}). \end{aligned} \tag{B6}$$

Here

$$
\begin{aligned}
a_{+1,+1} &= [(L - L' - l)(L + L' + l + 4p - 2)]^{\frac{1}{2}}, \\
a_{-1,+1} &= [(L - L' + l + 4)(L + L' - l + 4p - 6)]^{\frac{1}{2}}, \\
a_{\pm1,-1} &= [(L - L' \pm l + 2)(L + L' \mp l + 4p - 4)]^{\frac{1}{2}},
\end{aligned} \tag{B7}
$$

$$
\begin{aligned}
b_{+1,+1} &= [(L' + 2p - 2)(L' - L'' + l' + 4)(L' + L'' + l' + 4p - 6)]^{\frac{1}{2}}, \\
b_{+1,-1} &= [(L' + 2p - 2)(L' - L'' - l' + 2)(L' + L'' - l' + 4p - 8)]^{\frac{1}{2}}, \\
b_{-1,+1} &= [(L' + 2p - 4)(L' - L'' - l')(L' + L'' - l' + 4p - 6)]^{\frac{1}{2}}, \\
b_{-1,-1} &= [(L' + 2p - 4)(L' - L'' + l' + 2)(L' + L'' + l' + 4p - 8)]^{\frac{1}{2}},
\end{aligned} \tag{B8}
$$

$$
\begin{aligned}
c^{\pm}_{+1} &= [(l + 2)(l \pm m + |\bar{m}| + 2)(l \pm m - |\bar{m}| + 2)]^{\frac{1}{2}}, \\
c^{\pm}_{-1} &= [l(l \mp m + |\bar{m}|)(l \mp m - |\bar{m}|)]^{\frac{1}{2}}, \\
\tilde{c}^{\pm}_{+1} &= [(l + 2)(l + |m| \pm \bar{m} + 2)(l - |m| \pm \bar{m} + 2)]^{\frac{1}{2}}, \\
\tilde{c}^{\pm}_{-1} &= -[l(l + |m| \mp \bar{m})(l - |m| \mp \bar{m})]^{\frac{1}{2}}.
\end{aligned} \tag{B9}
$$

The coefficients $d^{\pm}_{\pm}$ and $\tilde{d}^{\pm}_{\pm}$ are the same as $c^{\pm}_{\pm}$ and $\tilde{c}^{\pm}_{\pm}$, respectively, if one substitutes l', m', and $\bar{m}'$ for l, m, and $\bar{m}$, respectively.

In the case of the representations $D^L_L(X^{4n-1})$ of $Sp(n)$, when

$$L = L' + l, \quad L' = L'' + l'$$

and

$$l = m - \bar{m}, \quad l' = m' - \bar{m}',$$

we have

$$a_{+1,+1} = b_{-1,+1} = c^{+}_{-1} = d^{+}_{-1} = \tilde{c}^{-}_{-1} = \tilde{d}^{-}_{-1} = 0. \tag{B10}$$

This reduces expressions (B5) and (B6) to the following form:

$$
\begin{aligned}
E_{\pm\mathbf{e}_p\pm\mathbf{e}_{p-1}} Y^{L,L';l,l'}_{m,m';\bar{m},\bar{m}'} &= 16[(l + 1)(l' + 1)(L' + 2p - 3)]^{-\frac{1}{2}} \\
&\times \{-a_{\pm1,\mp1} b_{\pm1,\pm1} \tilde{c}^{\pm}_{\mp1} d^{\pm}_{\pm1} Y^{L,L'\pm1;l\mp1,l'\pm1}_{m,m'\pm1,\bar{m}\pm1,\bar{m}'} + a_{\mp1,\pm1} b_{\mp1,\mp1} c^{\pm}_{\pm1} \tilde{d}^{\pm}_{\mp1} Y^{L,L'\mp1,l\pm1,l'\mp1}_{m\pm1,m',\bar{m},\bar{m}'\pm1}\},
\end{aligned} \tag{B11}
$$

$$
\begin{aligned}
E_{\pm\mathbf{e}_p\mp\mathbf{e}_{p-1}} Y^{L,L';l,l'}_{m,m';\bar{m},\bar{m}'} &= 16[(l + 1)(l' + 1)(L' + 2p - 3)]^{-\frac{1}{2}} \\
&\times \{\pm a_{\pm1,\mp1} b_{\pm1,\pm1} c^{\mp}_{\mp1} d^{\pm}_{\pm1} Y^{L,L'\pm1,l\mp1,l'\pm1}_{m\mp1,m'\pm1,\bar{m},\bar{m}'} \mp a_{\mp1\pm,1} b_{\mp1,\mp1} \tilde{c}^{\pm}_{\pm1} \tilde{d}^{\mp}_{\mp1} Y^{L,L'\mp1,l\pm1,l'\mp1}_{m,m';\bar{m}\mp1,\bar{m}'\pm1}\}.
\end{aligned} \tag{B12}
$$

The action of the invariant operator $\hat{I}^{(2)}_1$ connected with the $Sp(1)$ component of $Sp(1) \otimes Sp(n)$ is given by the formula

$$
\begin{aligned}
\hat{I}^{(2)}_1 Y^{L,L_s;l_s}_{M_n^+,\cdots,M_1^+;M_n^-,\cdots,M_1^-} &= \left[\sum_{p=1}^{n} l_p(l_p + 2) + 2 \sum_{1\le p<r\le n} M_p^- M_r^-\right] Y^{L,L_s;l_s}_{M_n^+,\cdots,M_1^+;M_n^-,\cdots,M_1^-} \\
&+ \sum_{1\le p<r\le n} [C_{pr} Y^{L,L_s;l_s}_{M_n^+,\cdots,M_1^+;M_n^-,\cdots,M_p^-+2,\cdots,M_r^--2,\cdots,M_1^-} \\
&+ C_{rp} Y^{L,L_s;l_s}_{M_n^+,\cdots,M_1^+;M_n^-,\cdots,M_p^--2,\cdots,M_r^-+2,\cdots,M_1^-}],
\end{aligned} \tag{B13}
$$

where

$$C_{pr} = [(l_p - M_p^-)(l_p + M_p^- + 2)(l_r + M_r^-)(l_r - M_r^- + 2)]^{\frac{1}{2}}. \tag{B14}$$

APPENDIX C: CALCULATION OF THE HIGHEST WEIGHT

In a representation D of a semisimple Lie group G of rank n, in which the matrices $D(H_i)$ of the Cartan subgroup generators H_i are diagonal, the eigenstates and eigenvalues of $D(H_i)$ are defined by

$$D(H_i)f = m_i f. \tag{C1}$$

The set of eigenvalues $\{m_i\}$ may be considered as a n-dimensional vector in the so-called weight space.[18] A weight $\mathbf{m}$ is called higher than another weight $\mathbf{m}'$ if the first nonvanishing component of $\mathbf{m} - \mathbf{m}'$ is a positive number. The weight $\mathbf{\Lambda}$, which is higher than any other weight in a given representation, is called the highest weight. As is well known,[19] the highest weight fully characterizes the irreducible representation of a compact group. We shall denote its components by Λ.

It has been proved by Cartan that there exist n

[18] R. E. Behrends *et al.*, Rev. Mod. Phys. 34, 1 (1962).

[19] E. B. Dynkin, Usp. Math. Nauk 2, 4(20) 59 (1947).

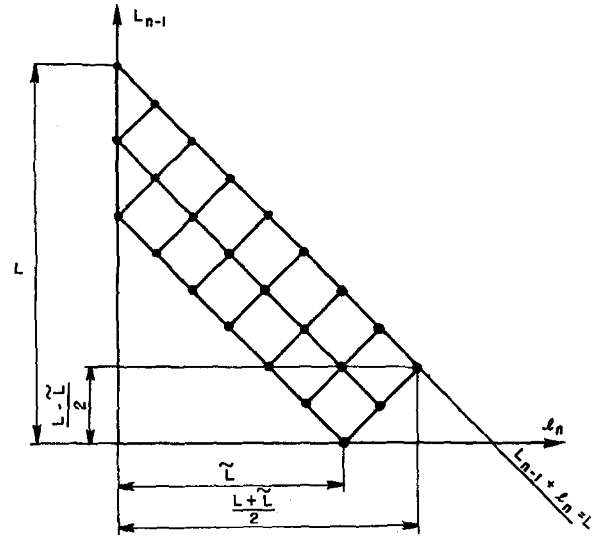

FIG. 5. The less degenerate representation of $Sp(n)$; decomposition with respect to the subgroup $Sp(1) \otimes Sp(n-1)$.

fundamental weights $\mathbf{\Lambda}^{(1)}, \cdots, \mathbf{\Lambda}^{(n)}$, and that any highest weight $\mathbf{\Lambda}$ is a linear combination

$$\mathbf{\Lambda} = \sum_{k=1}^{n} \lambda_k \mathbf{\Lambda}^{(k)}, \tag{C2}$$

where the λ_k are nonnegative integers. Moreover, these fundamental weights are uniquely defined by the root system of the algebra of G. From the system $\{\alpha^{(1)}, \cdots, \alpha^{(n)}\}$ of simple roots, we can calculate the Cartan matrix $\{A_{ij}\}$ by

$$A_{ij} = \frac{2(\boldsymbol{\alpha}^{(i)}, \boldsymbol{\alpha}^{(j)})}{(\boldsymbol{\alpha}^{(j)}, \boldsymbol{\alpha}^{(j)})}. \tag{C3}$$

Then the fundamental weights are given by

$$\mathbf{\Lambda}^{(k)} = \sum_{l=1}^{n} (A^{-1})_{kl} \boldsymbol{\alpha}^{(l)}, \tag{C4}$$

where A^{-1} is the matrix inverse to $\{A_{ij}\}$. Usually an irreducible representation of the group G is denoted by $D(\lambda_1, \cdots, \lambda_n)$. This notation appears throughout Secs. 3 and 4 and in Paper II.

Let us calculate the coefficients $\lambda_1, \cdots, \lambda_n$ for the representations $D_{\tilde{L}}^{L}$ of the group $Sp(n)$. The eigenvalue of the generator

$$H_n = i\left(\frac{\partial}{\partial \varphi_n} + \frac{\partial}{\partial \psi_n}\right) \tag{C5}$$

is clearly

$$-l_n \leq m_n + \bar{m}_n \leq l_n. \tag{C6}$$

Therefore the highest weight is the one with the highest value of l_n. From Fig. 5 we see that the highest possible value of l_n is

$$l_n = \frac{L + \tilde{L}}{2}. \tag{C7}$$

To this value corresponds

$$L_{n-1} = \frac{L - \tilde{L}}{2}. \tag{C8}$$

Hence there is a highest possible value of l_{n-1}, given by

$$l_{n-1} = \frac{L - \tilde{L}}{2}. \tag{C9}$$

But, when (C7) and (C9) hold simultaneously, it is easy to see that

$$l_{n-2} = \cdots = l_1 = 0. \tag{C10}$$

So we have

$$(\Lambda_1, \cdots, \Lambda_n) = \left(0, \cdots, 0, \frac{L - \tilde{L}}{2}, \frac{L + \tilde{L}}{2}\right). \tag{C11}$$

Now in the case of group $Sp(n)$ the Cartan matrix is given by

$$\{A_{ij}\} = \begin{vmatrix} 2 & -1 & \cdots & 0 & 0 & 0 \\ -1 & 2 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -2 & -2 \end{vmatrix} \tag{C12}$$

if we use the definition of $\alpha^{(i)}$ $(i = 1, \cdots, n)$ given in Appendix A (A16). Then the set of fundamental weights of $Sp(n)$ is

$$\begin{aligned}
\mathbf{\Lambda}^{(1)} &= \boldsymbol{\alpha}^{(1)} + \boldsymbol{\alpha}^{(2)} + \cdots + \boldsymbol{\alpha}^{(n-1)} + (\tfrac{1}{2})\boldsymbol{\alpha}^{(n)} = \mathbf{e}_n, \\
\mathbf{\Lambda}^{(2)} &= \boldsymbol{\alpha}^{(1)} + 2\boldsymbol{\alpha}^{(2)} + \cdots + 2\boldsymbol{\alpha}^{(n-1)} + \boldsymbol{\alpha}^{(n)}, \\
&= \mathbf{e}_{n-1} + \mathbf{e}_n, \\
&\cdot \\
&\cdot \\
&\cdot \\
\mathbf{\Lambda}^{(n-1)} &= \boldsymbol{\alpha}^{(1)} + 2\boldsymbol{\alpha}^{(2)} + \cdots + (n-1)\boldsymbol{\alpha}^{(n-1)} \\
&\quad + [(n-1)/2]\boldsymbol{\alpha}^{(n)} = \mathbf{e}_2 + \cdots + \mathbf{e}_n, \\
\mathbf{\Lambda}^{(n)} &= \boldsymbol{\alpha}^{(1)} + 2\boldsymbol{\alpha}^{(2)} + \cdots + (n-1)\boldsymbol{\alpha}^{(n-1)} \\
&\quad + (n/2)\boldsymbol{\alpha}^{(n)} = \mathbf{e}_1 + \cdots + \mathbf{e}_n.
\end{aligned} \tag{C13}$$

In the case of the representation D_M^L of $Sp(n)$ this gives

$$\begin{aligned}
\lambda_1 &= \frac{L + \tilde{L}}{2} - \frac{L - \tilde{L}}{2} = \tilde{L}, \\
\lambda_2 &= \frac{L - \tilde{L}}{2}, \\
\lambda_3 &= \cdots = \lambda_n = 0.
\end{aligned} \tag{C14}$$

Therefore, we have obtained the formula

$$D_{\tilde{L}}^{L} = D\left(\tilde{L}, \frac{L - \tilde{L}}{2}, 0, \cdots, 0\right), \tag{C15}$$

as indicated in Secs. 3 and 4.

Korteweg-de Vries Equation and Generalizations. I. A Remarkable Explicit Nonlinear Transformation*

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An explicit nonlinear transformation relating solutions of the Korteweg-de Vries equation and a similar nonlinear equation is presented. This transformation is generalized to solutions of a one-parameter family of similar nonlinear equations. A transformation is given which relates solutions of a "forced" Korteweg-de Vries equation to those of the Korteweg-de Vries equation.

1. INTRODUCTION

Interest in nonlinear dispersive wave equations has focused recently on the simplest model equation of this type, namely,

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

where subscripts denote partial differentiations. Korteweg and de Vries[1] first derived (1) (the KdV equation) in their study of long water waves in a (relatively shallow) channel. Recently, this equation has been derived in plasma physics[2,3] and in studies of anharmonic (nonlinear) lattices.[4,5] Existence and uniqueness of solutions of the KdV equation for appropriate initial and boundary conditions have recently been proved by Sjöberg.[6] The simplest modification of the nonlinear term in (1) leads to a similar equation

$$v_t + v^2 v_x + v_{xxx} = 0, \tag{2}$$

which also arises in the study of anharmonic lattices.[5]

The present paper is the first in a prospective series of works on properties and solutions of the KdV equation and its generalizations.[7]

I am privileged to write this first paper in the series, which presents a remarkable explicit nonlinear transformation between solutions of (1) and (2). Also, a transformation to an accelerating coordinate system is presented which relates solutions of (1) and a "forced" KdV equation. The second paper in the series[8] will discuss the existence of conservation laws and constants of motion for these equations. Also, it will show how the nonlinear transformation leads to associated eigenvalue problems. The third paper will show that the KdV equation governs small but finite perturbations from homogeneous equilibrium for a wide class of nonlinear dispersive systems. The fourth will show how the KdV equation and some generalizations can be viewed as Hamiltonian systems. The fifth paper will give a detailed discussion of polynomial conservation laws, including uniqueness and nonexistence proofs. The sixth paper in this series will consider the associated eigenvalue problems and will show how a study of them leads to exact general solution of the KdV equation. These papers will be referred to as I, II, III, IV, V, and VI.

2. TRANSFORMATION RELATING EQUATIONS (1) AND (2)

Equations (1) and (2) are particularly interesting, since they are exceptional among equations of the form

$$u_t + u^p u_x + u_{xxx} = 0, \quad p = 1, 2, 3, \cdots, \tag{3}$$

as the only ones possessing more than three "polynomial conservation laws" (not trivially equivalent; see II, Sec. 2. This result will be proved in V).

The similarity between (1) and (2), both in form and in possession of many polynomial conservation laws (see II), suggested that their solutions might be intimately related. A detailed comparison of these laws led to the discovery that if v satisfies (2), then u, defined by

$$u \equiv v^2 \pm (-6)^{\frac{1}{2}} v_x, \tag{4}$$

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† Present address: Courant Institute of Mathematical Sciences, New York University, New York, N.Y.

[1] D. J. Korteweg and G. de Vries, Phil. Mag. 39, 422 (1895).

[2] C. S. Gardner and G. K. Morikawa, Courant Institute of Mathematical Sciences Report No. NYO-9082, 1 May 1960 (unpublished).

[3] H. Washimi and T. Taniuti, Phys. Rev. Letters 17, 996 (1966).

[4] M. D. Kruskal, "Asymptotology in Numerical Computation: Progress and Plans on the Fermi-Pasta-Ulam Problem" in *Proceedings of the IBM Scientific Computing Symposium on Large-Scale Problems in Physics* (IBM Data Processing Division, White Plains, N.Y., 1965), p. 43.

[5] N. J. Zabusky, "A Synergetic Approach to Problems of Nonlinear Dispersive Wave Propagation and Interaction" in *Proceedings of the Symposium on Nonlinear Partial Differential Equations*, W. Ames, Ed. (Academic Press Inc., New York, 1967), p. 223.

[6] A. Sjöberg, "On the Korteweg-de Vries Equation, Existence and Uniqueness" (Uppsala University, Department of Computer Sciences, Uppsala, Sweden, 1967) (unpublished).

[7] This program of research has been conducted mainly by C. S. Gardner, J. M. Greene, M. D. Kruskal, C.-H. Su, and the author at the Plasma Physics Laboratory, Princeton University, Princeton, N.J., and by N. J. Zabusky at the Bell Telephone Laboratories, Inc., Whippany, N.J. Specific individual contributions will be reflected in the authorship of the various papers.

[8] J. Math. Phys. 9, 1204 (1968), following paper.

satisfies (1). By explicit calculation, in fact,

$$u_t + uu_x + u_{xxx} \equiv \left(2v \pm (-6)^{\frac{1}{2}} \frac{\partial}{\partial x}\right)(v_t + v^2 v_x + v_{xxx}). \quad (5)$$

The presence of the operator $(2v \pm (-6)^{\frac{1}{2}}\partial/\partial x)$ hinders us from concluding inversely that if u satisfies (1), then any solution v of the Riccati equation (4) is a solution of (2).

The reader need not be concerned about the occurrence of the imaginary coefficient in (4). It is an historic accident that we chose to study (1) and (2) with the signs of the terms as given. For (1), the particular choice of signs is unimportant, since appropriate changes of sign of the variables yield transformations between any two possibilities. (See V for transformation properties of the KdV equation.) However, for (2), the relative sign of the last two terms is invariant to such transformations, but can be reversed by the substitution $v \to iv$. We could have confined our discussion here to real solutions by considering two versions of (2), one with like and one with unlike signs.

The transformation takes (2) with cubic nonlinearity into the quadratically nonlinear KdV equation (1). It is rare and surprising to find a transformation between two simple nonlinear partial differential equations of independent interest. One is reminded of the Hopf–Cole transformation[9,10] of the quadratically nonlinear Burgers equation into the linear heat conduction (diffusion) equation. A number of investigators (including us) have attempted unsuccessfully to find a similar simple linearizing transformation for the KdV equation, but a complicated one will be given in VI.

3. A GENERALIZATION

A generalization[11] of the transformation (4), which in II is used to prove the existence of infinitely many conservation laws, is that a one-parameter family of nonlinear equations similar to (1) and (2), but containing both types of nonlinear terms simultaneously, can be transformed into (1). Noting that (1) is invariant to Galilean transformation (again, see V), whereas (2) is not, we define

$$t' \equiv t, \quad x' \equiv x - \frac{3}{2\epsilon^2}\, t, \quad (6)$$

$$u(x, t) \equiv u'(x', t') + \frac{3}{2\epsilon^2}, \quad (7)$$

$$v(x, t) \equiv \frac{\epsilon}{\sqrt{6}}\, w(x', t') + \frac{\sqrt{6}}{2\epsilon}, \quad (8)$$

where the specific dependence on the arbitrary parameter ϵ has been chosen for convenience in II. Then (1) remains invariant, of course, but (2) (dropping the primes) becomes

$$w_t + (w + \tfrac{1}{6}\epsilon^2 w^2) w_x + w_{xxx} = 0, \quad (9)$$

and (4) (with the plus sign) becomes

$$u \equiv w + i\epsilon w_x + \tfrac{1}{6}\epsilon^2 w^2. \quad (10)$$

We observe that (9) reduces to (1) for $\epsilon = 0$, and after the rescaling $w' \equiv (\epsilon/\sqrt{6})w$ it reduces to (2) for $\epsilon \to \infty$.

4. TRANSFORMATION TO ACCELERATING COORDINATE SYSTEM

The KdV equation (1) can be generalized by adding a time-dependent "forcing term," and for convenience we write it as

$$u_t + uu_x + u_{xxx} = y_{tt}, \quad (11)$$

where we assume that $y_{tt} = y_{tt}(t)$ is a known function. This equation arises in the study of ion-acoustic waves.[3,12] It also arises in a study of the propagation of electrostatic waves through an ion sheath where $y_{tt} = 1$.[13]

We now give a transformation which reduces the "forced" KdV equation (11) to the KdV equation (1). This transformation is also applicable to more general equations (e.g., the Burgers equation) where u_{xxx} is replaced by any arbitrary function of x derivatives of u not depending on either u itself or explicitly on x or t. Define new variables

$$t' \equiv t, \quad x' \equiv x - y(t), \quad (12)$$

$$u(x, t) \equiv u'(x', t') + y_{t'}(t'). \quad (13)$$

Direct substitution of this transformation into (11) shows that the KdV equation (1) is indeed obtained for the primed variables. We note a strong similarity to the Galilean transformation (6) and (7).

The physical interpretation of this transformation is clear. The quantity $y(t)$ represents a time-dependent translation of the x axis, and, therefore, the forcing

[9] E. Hopf, Commun. Pure Appl. Math. **3**, 201 (1950).
[10] J. D. Cole, Quart. Appl. Math. **9**, 225 (1951).
[11] I am grateful to C. S. Gardner for this generalization.
[12] This equation does not appear in the paper by Washimi and Taniuti because they failed to account for the arbitrary integration "constant" $y_t(t)$ in their solution relating the first-order electron density and ion velocity. This has been corrected in a recent paper, T. Taniuti and C.-C. Wei, J. Phys. Soc. Japan **24**, 941 (1968).
[13] S. H. Lam and C. Berman, Department of Aerospace and Mechanical Sciences, Princeton University (private communication).

term in (11) may be interpreted as due solely to an acceleration of the x axis.

Since we have assumed only that $y_{tt}(t)$ is known, we have the freedom to set $y(0) = y_t(0) = 0$. With this information the transformation becomes particularly useful, since the initial values for the two equations are identical:

$$u(x, 0) = u'(x', 0) = u'(x, 0). \tag{14}$$

Therefore, if the solution of the KdV equation (1) is known, then the full solution is obtained from (12) and (13) with only two simple quadratures to obtain $y_t(t)$ and $y(t)$.

Note added in proof: The transformation (12) and (13) was used by Moore[14] for studying the viscous boundary layer on an accelerating semi-infinite flat plate. I wish to thank H.-H. Chiu of the Department of Aeronautics and Astronautics at New York University for this reference.

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I wish to thank C. S. Gardner and M. D. Kruskal for helpful discussions.

[14] F. K. Moore, "Unsteady Laminar Boundary-Layer Flow," National Advisory Committee for Aeronautics, Tech. Note 2471, September 1951.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 9, NUMBER 8 AUGUST 1968

Korteweg–de Vries Equation and Generalizations. II. Existence of Conservation Laws and Constants of Motion*

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(Received 8 October 1967)

With extensive use of the nonlinear transformations presented in Paper I of the series, a variety of conservation laws and constants of motion are derived for the Korteweg–de Vries and related equations. A striking connection with the Sturm–Liouville eigenvalue problem is exploited.

1. INTRODUCTION

In this second paper of the series on the properties and solutions of the KdV equation, $u_t + uu_x + u_{xxx} = 0$ and its generalizations, we present our current body of knowledge on the existence of conservation laws and of constants of motion (i.e., "temporal invariants") for the KdV equation (I.1) and two similar nonlinear equations (I.2) and (I.9) given in the first paper of this series,[1] referred to as I. The present Paper II is meant to be read in conjunction with I, where nonlinear transformations relating solutions of (I.1) to those of (I.2) and (I.9) are given. (References to physical applications are also given there.) Most of the results on the conservation laws and the constants of motion are based on, or are in some way related to, these transformations.

A *conservation law* associated with an equation such as (I.1) is expressed by an equation of the form

$$T_t + X_x = 0, \tag{1}$$

where T, the *conserved density*, and $-X$, the *flux* of T, are functionals of u. If T is a *local* functional of u, i.e., if the value of T at any x depends only on the values of u in an arbitrarily small neighborhood of x, then T is a *local conserved density*; if X is also local, then (1) is a *local conservation law*. In particular, if T is a polynomial in u and its x derivatives and not dependent explicitly on x or t, then we call T a *polynomial conserved density*; if X is also such a polynomial, we call (1) a *polynomial conservation law*. [We need never allow for dependence on t derivatives of u, since (I.1) permits them to be eliminated in favor of x derivatives; similarly with the other such equations we deal with.] In Sec. 2 we present a number of polynomial conservation laws which have been found explicitly, and in Sec. 3 we prove that there are infinitely many of them for each of (I.1), (I.2), and (I.9).

There is a close relationship between constants (of motion) and conservation laws. For example, if one

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[1] R. M. Miura, J. Math. Phys. 9, 1202 (1968), preceding paper.

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[1] R. M. Miura, J. Math. Phys. 9, 1202 (1968), preceding paper.

assumes either that u is periodic in x or that u and its x derivatives vanish sufficiently rapidly at the (finite or infinite) ends of some interval, each polynomial conservation law (1) immediately yields a *constant of local conservation type*

$$I \equiv \int T\,dx. \tag{2}$$

(Throughout this paper integrals are to be taken as complete.) An example of a constant of "nonlocal conservation type" occurs in Sec. 5. However, we can also derive constants without use of conservation laws (e.g., the discrete eigenvalues discussed in Sec. 4), and, on the other hand, conservation laws yield constants of conservation type only under special conditions.

In Sec. 4 we show that the nonlinear transformation (I.4) leads naturally to a Sturm–Liouville equation (or time-independent Schrödinger equation). The discrete eigenvalues turn out to be constants of motion. Still another family of constants arises from a study of the time-evolution equations for the eigenfunctions, as shown in Sec. 5. Finally, in Sec. 6 we give a single constant of local conservation type with conserved density depending explicitly on x and t as well as on u.

2. THE EXPLICITLY KNOWN POLYNOMIAL CONSERVED DENSITIES

For a polynomial conservation law of (I.1) or (I.2), T and X are each a finite sum of terms of the form $u_0^{a_0}u_1^{a_1}\cdots u_l^{a_l}$, where $u_j \equiv \partial^j u/\partial x^j$ and the a_j are nonnegative integers. For each such term we define a *rank* r. In dealing with (I.1) the rank is the sum of the number of factors u_j and half the number of x differentiations,

$$r_1 \equiv \sum_{j=0}^{l} (1 + \tfrac{1}{2}j)a_j, \tag{3}$$

as is consistent with the scaling properties of the last two terms of the equation. In dealing with (I.2) the rank is

$$r_2 \equiv \tfrac{1}{2}\sum_{j=0}^{l} (1 + j)a_j, \tag{4}$$

similarly consistent. A *polynomial of rank* r is one whose terms are all of rank r. Since any x derivative is trivially a conserved density, two polynomials which differ by an x derivative will be called *equivalent*. Any polynomial conserved density for (I.1) or (I.2) can be uniquely expressed as a sum of polynomials of differing ranks. It is easily seen that these polynomials individually are conserved densities, since t differentiation increases the rank by $\tfrac{3}{2}$. In this section we prove that, for both (I.1) and (I.2), there is a polynomial conservation law with nontrivial conserved density T_r of each positive integral rank r (and corresponding flux X_r of rank $r + 1$).

Equation (I.1) can itself be expressed as a polynomial conservation law with $r = 1$, and multiplication by u yields one with $r = 2$. These are obvious, and in the usual applications correspond physically to conservation of momentum and energy. A polynomial conserved density with $r = 3$ was found by Whitham,[2] two more with $r = 4$ and 5 were found by Kruskal and Zabusky,[3] and we have explicitly computed five additional ones of consecutive rank. (A systematic method for such computations will be given in V.) For (I.2), similarly, we have explicitly computed polynomial conserved densities with $r = \tfrac{1}{2}$, 1, 2, 3, 4, and 5. These polynomial conserved densities are all the ones we know explicitly, and we record them here, together with the corresponding polynomial fluxes as far as we have calculated them. The freedom to add x derivatives has been utilized to write the conserved densities in a canonical form where the highest-derivative factor (if any) in each term occurs at least squared (as explained in detail in V), so the following formulas are unique up to multiplication by a constant.

For Eq. (I.1) we have

$$T_1 = u_0, \tag{5a}$$
$$X_1 = \tfrac{1}{2}u_0^2 + u_2, \tag{5b}$$

$$T_2 = \tfrac{1}{2}u_0^2, \tag{6a}$$
$$X_2 = \tfrac{1}{3}u_0^3 + u_0u_2 - \tfrac{1}{2}u_1^2, \tag{6b}$$

$$T_3 = \tfrac{1}{3}u_0^3 - u_1^2, \tag{7a}$$
$$X_3 = \tfrac{1}{4}u_0^4 + u_0^2u_2 - 2u_0u_1^2 - 2u_1u_3 + u_2^2, \tag{7b}$$

$$T_4 = \tfrac{1}{4}u_0^4 - 3u_0u_1^2 + \tfrac{9}{5}u_2^2, \tag{8a}$$
$$\begin{aligned} X_4 = {} & \tfrac{1}{5}u_0^5 + u_0^3u_2 - \tfrac{9}{2}u_0^2u_1^2 + \tfrac{24}{5}u_0u_2^2 - 6u_0u_1u_3 \\ & + 3u_1^2u_2 + \tfrac{18}{5}u_2u_4 - \tfrac{9}{5}u_3^2, \end{aligned} \tag{8b}$$

$$T_5 = \tfrac{1}{5}u_0^5 - 6u_0^2u_1^2 + \tfrac{36}{5}u_0u_2^2 - \tfrac{108}{35}u_3^2, \tag{9a}$$

$$\begin{aligned} X_5 = {} & \tfrac{1}{6}u_0^6 + u_0^4u_2 - 8u_0^3u_1^2 + \tfrac{66}{5}u_0^2u_2^2 - 12u_0^2u_1u_3 \\ & + 12u_0u_1^2u_2 - 3u_1^4 + \tfrac{72}{5}u_0u_2u_4 - \tfrac{72}{7}u_0u_3^2 \\ & - \tfrac{72}{5}u_1u_2u_3 + \tfrac{36}{35}u_2^3 - \tfrac{216}{35}u_3u_5 + \tfrac{108}{35}u_4^2, \end{aligned} \tag{9b}$$

$$\begin{aligned} T_6 = {} & \tfrac{1}{6}u_0^6 - 10u_0^3u_1^2 + 18u_0^2u_2^2 - 5u_1^4 - \tfrac{108}{7}u_0u_3^2 \\ & + \tfrac{120}{7}u_2^3 + \tfrac{36}{7}u_4^2, \end{aligned} \tag{10a}$$

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[2] G. B. Whitham, Proc. Roy. Soc. (London) **A283**, 238 (1965).

[3] N. J. Zabusky, "A Synergetic Approach to Problems of Nonlinear Dispersive Wave Propagation and Interaction" in *Proceedings of the Symposium on Nonlinear Partial Differential Equations*, W. Ames, Ed. (Academic Press Inc., New York, 1967), p. 223.

$$
\begin{aligned}
X_6 = {} & \tfrac{1}{7}u_0^7 + u_0^5u_2 - \tfrac{25}{2}u_0^4u_1^2 + 28u_0^3u_2^2 - 20u_0^3u_1u_3 \\
& + 30u_0^2u_1^2u_2 - 20u_0u_1^4 + 36u_0^2u_2u_4 - \tfrac{234}{7}u_0^2u_3^2 \\
& + \tfrac{156}{7}u_0u_2^3 - 72u_0u_1u_2u_3 + 66u_1^2u_2^2 - 20u_1^3u_3 \\
& - \tfrac{216}{7}u_0u_3u_5 - \tfrac{144}{7}u_0u_4^2 + \tfrac{360}{7}u_2^2u_4 - \tfrac{108}{7}u_2u_3^2 \\
& + \tfrac{216}{7}u_1u_3u_4 + \tfrac{72}{7}u_4u_6 - \tfrac{36}{7}u_5^2,
\end{aligned} \tag{10b}
$$

$$
\begin{aligned}
T_7 = {} & \tfrac{1}{7}u_0^7 - 15u_0^4u_1^2 + 36u_0^3u_2^2 - 30u_0u_1^4 - \tfrac{324}{7}u_0^2u_3^2 \\
& + \tfrac{720}{7}u_0u_2^3 + 108u_1^2u_2^2 + \tfrac{216}{7}u_0u_4^2 \\
& - \tfrac{1080}{7}u_2u_3^2 - \tfrac{648}{77}u_5^2,
\end{aligned} \tag{11a}
$$

$$
\begin{aligned}
X_7 = {} & \tfrac{1}{8}u_0^8 + u_0^6u_2 - 18u_0^5u_1^2 + 51u_0^4u_2^2 - 30u_0^4u_1u_3 \\
& + 60u_0^3u_1^2u_2 - 75u_0^2u_1^4 + 72u_0^3u_2u_4 - \tfrac{576}{7}u_0^3u_3^2 \\
& + \tfrac{828}{7}u_0^2u_2^3 - 216u_0^2u_1u_2u_3 + 504u_0u_1^2u_2^2 \\
& - 120u_0u_1^3u_3 + 90u_1^4u_2 - \tfrac{648}{7}u_0^2u_3u_5 \\
& + \tfrac{540}{7}u_0^2u_4^2 - \tfrac{1728}{7}u_0u_2u_3^2 + \tfrac{1296}{7}u_0u_1u_3u_4 \\
& - \tfrac{1404}{7}u_1^2u_3^2 + \tfrac{2160}{7}u_0u_2^2u_4 - \tfrac{522}{7}u_2^4 \\
& - \tfrac{3672}{7}u_1u_2^2u_3 + 216u_1^2u_2u_4 + \tfrac{432}{7}u_0u_4u_6 \\
& - \tfrac{432}{11}u_0u_5^2 - \tfrac{432}{7}u_1u_4u_5 + \tfrac{648}{11}u_2u_4^2 - \tfrac{1080}{77}u_3^2u_4 \\
& - \tfrac{2160}{7}u_2u_3u_5 - \tfrac{1296}{77}u_5u_7 + \tfrac{648}{77}u_6^2,
\end{aligned} \tag{11b}
$$

$$
\begin{aligned}
T_8 = {} & \tfrac{1}{8}u_0^8 - 21u_0^5u_1^2 + 63u_0^4u_2^2 - 105u_0^2u_1^4 - 108u_0^3u_3^2 \\
& + 360u_0^2u_2^3 + 756u_0u_1^2u_2^2 + 108u_0^2u_4^2 - 324u_1^2u_3^2 \\
& - 1080u_0u_2u_3^2 + 378u_2^4 - \tfrac{648}{11}u_0u_5^2 \\
& + \tfrac{4536}{11}u_2u_4^2 + \tfrac{1944}{143}u_6^2,
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
T_9 = {} & \tfrac{1}{9}u_0^9 - 28u_0^6u_1^2 + \tfrac{504}{5}u_0^5u_2^2 - 280u_0^3u_1^4 - 216u_0^4u_3^2 \\
& + 960u_0^3u_2^3 + 3024u_0^2u_1^2u_2^2 - 168u_1^6 + 288u_0^3u_4^2 \\
& - 4320u_0^2u_2u_3^2 - 2592u_0u_1^2u_3^2 + 3024u_0u_2^4 \\
& + \tfrac{26496}{5}u_1^2u_2^3 - \tfrac{2592}{11}u_0^2u_5^2 + \tfrac{36288}{11}u_0u_2u_4^2 \\
& + 864u_1^2u_4^2 - \tfrac{169344}{55}u_1u_3^3 - \tfrac{545184}{55}u_2^2u_3^2 \\
& + \tfrac{15552}{143}u_0u_6^2 - \tfrac{145152}{143}u_2u_5^2 + \tfrac{653184}{715}u_4^3 \\
& - \tfrac{15552}{715}u_7^2,
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
T_{10} = {} & \tfrac{1}{10}u_0^{10} - 36u_0^7u_1^2 - 630u_0^4u_1^4 + \tfrac{756}{5}u_0^6u_2^2 \\
& - 1512u_0u_1^6 + 2160u_0^4u_2^3 + 9072u_0^3u_1^2u_2^2 \\
& - \tfrac{1944}{5}u_0^5u_3^2 + 13608u_0^2u_2^4 + \tfrac{238464}{5}u_0u_1^2u_2^3 \\
& + 13608u_1^4u_2^2 - 12960u_0^3u_2u_3^2 - 11664u_0^2u_1^2u_3^2 \\
& + 648u_0^4u_4^2 + \tfrac{178848}{11}u_2^5 - \tfrac{1524096}{55}u_0u_1u_3^3 \\
& - \tfrac{4906656}{55}u_0u_2^2u_3^2 - \tfrac{334368}{5}u_1^2u_2u_3^2 \\
& + \tfrac{163296}{11}u_0^2u_2u_4^2 + 7776u_0u_1^2u_4^2 - \tfrac{7776}{11}u_0^3u_5^2 \\
& - \tfrac{11045808}{715}u_3^4 + \tfrac{5878656}{715}u_0u_4^3 \\
& + \tfrac{22208256}{715}u_1u_3u_4^2 + \tfrac{26570592}{715}u_2^2u_4^2 \\
& - \tfrac{1306368}{143}u_0u_2u_5^2 - \tfrac{23328}{11}u_1^2u_5^2 + \tfrac{69984}{143}u_0^2u_6^2 \\
& - \tfrac{5878656}{715}u_4u_5^2 + \tfrac{1679616}{715}u_2u_6^2 \\
& - \tfrac{139968}{715}u_0u_7^2 + \tfrac{419904}{12155}u_8^2.
\end{aligned} \tag{14}
$$

For Eq. (I.2) we have

$$T_{\frac12} = v_0, \tag{15a}$$

$$X_{\frac12} = \tfrac13 v_0^3 + v_2, \tag{15b}$$

$$T_1 = \tfrac12 v_0^2, \tag{16a}$$

$$X_1 = \tfrac14 v_0^4 + v_0v_2 - \tfrac12 v_1^2, \tag{16b}$$

$$T_2 = \tfrac14 v_0^4 - \tfrac32 v_1^2, \tag{17a}$$

$$X_2 = \tfrac16 v_0^6 + v_0^3v_2 - 3v_0^2v_1^2 - 3v_1v_3 + \tfrac32 v_2^2, \tag{17b}$$

$$T_3 = \tfrac16 v_0^6 - 5v_0^2v_1^2 + 3v_2^2, \tag{18}$$

$$T_4 = \tfrac18 v_0^8 - \tfrac{21}{2}v_0^4v_1^2 + \tfrac{63}{5}v_0^2v_2^2 - \tfrac{63}{10}v_1^4 - \tfrac{27}{5}v_3^2, \tag{19}$$

$$
\begin{aligned}
T_5 = {} & \tfrac{1}{10}v_0^{10} - 18v_0^6v_1^2 + \tfrac{162}{5}v_0^4v_2^2 - \tfrac{342}{5}v_0^2v_1^4 - \tfrac{972}{35}v_0^2v_3^2 \\
& + \tfrac{432}{7}v_0v_2^3 + \tfrac{5508}{35}v_1^2v_2^2 + \tfrac{324}{35}v_4^2.
\end{aligned} \tag{20}
$$

3. EXISTENCE OF INFINITE SEQUENCES OF POLYNOMIAL CONSERVED DENSITIES

The orderliness of these conservation laws led us quite early to conjecture that there exists one of every positive rank, both for (I.1) and for (I.2). The rapid proliferation of terms with increasing rank, however, makes a straightforward inductive argument extremely difficult. We prove the conjecture in this section by finding a recursion algorithm for constructing the conserved densities; the corresponding fluxes are then obtained in terms of them. In V we will discuss these polynomial conserved densities in more detail and prove that they are unique up to addition of an x derivative and multiplication by a constant. Also we will prove there that nontrivial polynomial conserved densities of half-integral rank do not exist for (I.1), nor for (I.2) except with $r_2 = \frac12$.

By solving (I.10) recursively, w can be expressed as a formal series of increasing nonnegative integral powers of ϵ, the coefficient of ϵ^n being a polynomial in u and its derivatives of rank $1 + \frac12 n$. (Informally, ϵ may be thought of as a small expansion parameter.) Equation (I.10) transforms (I.1) into

$$0 = \left(1 + i\epsilon\frac{\partial}{\partial x} + \tfrac13\epsilon^2 w\right)[w_t + (w + \tfrac16\epsilon^2w^2)w_x + w_{xxx}], \tag{21}$$

and, since we are dealing with formal series, obviously the expression in square brackets must itself vanish (to all orders), which gives (I.9) formally. Thus we obtain a conservation law with

$$T = w, \quad X = \tfrac12 w^2 + \tfrac{1}{18}\epsilon^2 w^3 + w_{xx}. \tag{22}$$

If we now substitute the series for w into these, we obtain a formal-series conservation law for (I.1). Then, since (I.1) is independent of ϵ, for each n, the coefficients of ϵ^n in T and X constitute a conservation law for (I.1). [It actually suffices to generate the conserved densities, since the corresponding fluxes are always obtainable by antidifferentiation from (1).]

It remains to demonstrate that the conserved density for each even n is nontrivial, i.e., not an x derivative (unlike the odd ones, as we shall see shortly). To do this we show that each such density contains a term (with nonzero coefficient) which is purely a power of u, as will suffice since such terms can never arise from differentiation. Now, by the last remark, all such terms must be generated by recursion from (I.10) with the w_x term omitted. This equation can even be solved explicitly, yielding $-(3/\epsilon^2)[1-(1+\frac{2}{3}\epsilon^2 u)^{\frac{1}{2}}]$ as the sum of all terms not depending on the derivatives of u. But when this is expanded, evidently every nonnegative even power of ϵ actually appears. This concludes the proof of the conjecture.

We now show that the coefficients of the odd powers of ϵ, which comprise the imaginary terms, are all x derivatives, so that these conserved densities of half-integral rank are trivial. Assume that u is real and define

$$w \equiv y + iz, \tag{23}$$

where y and z are real. From (I.10) it is obvious that y contains only even and z only odd powers of ϵ. Now (I.10) becomes

$$u \equiv y - \epsilon z_x + \tfrac{1}{6}\epsilon^2(y^2 - z^2) + i(z + \epsilon y_x + \tfrac{1}{3}\epsilon^2 yz). \tag{24}$$

The imaginary part of this equation may be written

$$z = -\frac{3}{\epsilon}[\ln(1 + \tfrac{1}{3}\epsilon^2 y)]_x; \tag{25}$$

therefore z is an x derivative to every order in ϵ, as claimed. Incidentally, by using (25) to eliminate z in the real part of (24), we can obtain a recursion formula which generates the nontrivial conserved densities y but, unlike (I.10), "skips over" the intervening trivial ones.

Although (I.10) is a simple recursion formula for the conserved densities, it would be quite awkward to write an explicit formula for the conserved density of rank r obtained from it. A much simpler and more elegant formula for the conserved density of rank r will be obtained in V.

Having established the existence of an infinite sequence of (nontrivial) conserved densities for (I.1), from (I.4) we obtain such a sequence for (I.2), and from (1.10) for (I.9). In fact, it was a correspondence noticed between conserved densities of (I.1) and (I.2) which originally suggested the transformation (I.4). Under appropriate boundary conditions, we immediately have the existence of infinitely many constants of motion given by (2).

It may be remarked that if we had used, instead of (22), one of the just mentioned higher polynomial conserved densities of the w equation (I.9), e.g., $T = \frac{1}{2}w^2$, we would also have obtained an infinite sequence of conserved densities for (I.1). But the results would necessarily be included in the previous ones, because of the uniqueness to be proved in V.

We note that th single conserved density w for (I.9) yielded infinitely many conserved densities for (I.1). However, the inverse does not hold. One reason for this is that u as expressed in terms of w by (I.10) is a finite series, and another is that, even if it were infinite, because (21) depends on ϵ we could not conclude that the coefficient of each power of ϵ would be a conserved density.

4. ASSOCIATED EIGENVALUE PROBLEM

Viewing (I.4) (with plus sign) as a Riccati equation for v, we introduce the usual linearizing change of variables

$$v \equiv (-6)^{\frac{1}{2}}\frac{\psi_x}{\psi}, \tag{26}$$

transforming it into

$$u = -6\frac{\psi_{xx}}{\psi}. \tag{27}$$

For most of this section, we consider the case that all functions involved (i.e., u, v, ψ) are periodic with a common period. If (27) is interpreted as an equation for ψ, then for almost any u there is no (periodic) solution. We may, however, take advantage of the Galilean invariance of (I.1) and shift u by a constant. We therefore replace u in (27) by $u - \lambda$ and obtain

$$\psi_{xx} + \tfrac{1}{6}(u - \lambda)\psi = 0, \tag{28}$$

which is the well-known Sturm–Liouville equation.

We briefly summarize some familiar properties of the eigenvalues λ and corresponding eigenfunctions ψ for the periodic Sturm–Liouville eigenvalue problem[4]:

(i) There is a denumerable infinity of eigenvalues, all real, satisfying $\lambda_0 > \lambda_1 \geq \lambda_2 > \lambda_3 \geq \lambda_4 > \lambda_5 \geq \lambda_6 > \cdots$ and approaching $-\infty$;

(ii) The corresponding (real) eigenfunctions $\psi^{(n)}$ form a complete orthogonal system;

(iii) The zeros of $\psi^{(n)}$ are all simple, and the number of them per period is n for n even and $n + 1$ for n odd;

(iv) A function ψ' is an eigenfunction if and only

[4] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations* (McGraw-Hill Book Co., Inc., New York, 1955).

if it makes the functional

$$\Lambda\{\psi\} \equiv \int (u\psi^2 - 6\psi_x^2)\, dx \Big/ \int \psi^2\, dx \qquad (29)$$

stationary, and the corresponding eigenvalue λ' is $\Lambda\{\psi'\}$.

If we allow u to evolve according to the KdV equation, then it is natural to ask how λ and ψ evolve. Eliminating u by (28) from the KdV equation, the result may be written

$$0 = \lambda_t - \frac{6}{\psi^2}\frac{\partial}{\partial x}\left[\left(\psi\frac{\partial}{\partial x} - \psi_x\right) \times \left(\psi_t - 3\frac{\psi_x\psi_{xx}}{\psi} + \psi_{xxx} + \lambda\psi_x\right)\right]. \qquad (30)$$

Multiplying by ψ^2 and integrating over the period gives

$$\lambda_t = 0; \qquad (31)$$

thus each eigenvalue is a constant of motion! These new, infinitely numerous constants appear not to be associated with any nontrivial conservation laws, i.e., with nontrivial conserved densities (see Sec. 2).

Appropriately integrating (30) twice gives for the last expression in parentheses, rewritten again by (28) to remove even the apparency of a possible singularity when $\psi = 0$,

$$\psi_t + \tfrac{1}{2}(u + \lambda)\psi_x + \psi_{xxx} = \alpha\psi + \beta\psi\int^x \psi^{-2}\, dx, \qquad (32)$$

where $\alpha(t)$ and $\beta(t)$ are the constants of integration. (For the interpretation of the integral when ψ has zeros, see Sec. 5.) By periodicity, $\beta = 0$ unless $\int \psi^{-2}\, dx = 0$, this latter being the condition on ψ that (28) have a second linearly independent periodic solution. Thus ψ_t is determined as nearly uniquely as could be expected: the freedom to normalize ψ arbitrarily at different times is represented by the choice of α, and to add in the second eigenfunction, when λ is a double eigenvalue by the choice of β.

Conversely, we can verify that if u and ψ evolve according to (I.1) and (32), then (28) remains satisfied. For denoting the left side of (28) by Q, straightforward calculation leads to

$$Q_t + \tfrac{1}{2}[(u + \lambda)Q]_x + Q_{xxx} = 0. \qquad (33)$$

If $Q = 0$ initially, it remains so for all time, as shown by an "energy" argument in which (33) is multiplied by Q and integrated, and $\int Q^2\, dx$ is shown by simple estimation to grow no faster than exponentially in time.

It is interesting to examine the eigenfunctions for large (negative) λ by means of the familiar WKB formalism. We set

$$\psi \approx \mathrm{Re}\left[B \exp\left\{(\lambda/6)^{\frac{1}{2}}\int^x A\, dx\right\}\right], \qquad (34)$$

where A and B are asymptotic series in nonpositive integral powers of $\lambda^{\frac{1}{2}}$, all coefficients of which have the common period in x; the factor B will be chosen later for convenience, but we could set $B = 1$ without loss of generality. We require the expression in brackets to satisfy (28), obtaining a condition which, when solved for the A in the leading terms, yields

$$A = \left[1 - (6/\lambda)^{\frac{1}{2}}\left(A_x + \frac{2B_x}{B}A\right) - \frac{1}{\lambda}\left(u + \frac{6B_{xx}}{B}\right)\right]^{\frac{1}{2}} \qquad (35)$$

as a recursion formula for A. Periodicity of (34) requires

$$\int A\, dx = 2\pi N(-6/\lambda)^{\frac{1}{2}}, \qquad (36)$$

N being the (large) number of zeros of ψ in a period. [Taking the imaginary instead of the real part in (34) gives the other eigenfunction with N zeros; thus the complex version of (34) represents both $\psi^{(N-1)}$ and $\psi^{(N)}$ at once, and $\lambda_{N-1} = \lambda_N$ to all orders as $N \to \infty$.] Seen to be even more effective than $B = 1$ is the choice $B = A^{-\frac{1}{2}}$, which eliminates the $\lambda^{-\frac{1}{2}}$ term in (35). With either choice, the coefficient of any power of λ in A is evidently a polynomial in u and its derivatives, and is a conserved density by (36). With $B = A^{-\frac{1}{2}}$, in fact, the simple replacements $A \equiv 1 + \frac{1}{3}\epsilon^2 y$ and $\lambda \equiv -\frac{3}{2}\epsilon^{-2}$ transform (35) into the recursion formula for the nontrivial polynomial conserved densities mentioned after (25). The derivation here is more general, however, since it shows that if the evolution of u in (28) is governed by any equation whatsoever which leaves the eigenvalues invariant, then that equation possesses all the same polynomial conserved densities as the KdV equation. Lax[5] has initiated a search for such equations and found several, and Lenard[6] has recently derived an infinite sequence of such equations.

Turning now to the case of the infinite interval with u vanishing sufficiently rapidly as $x \to \pm\infty$, we have the time-independent Schrödinger equation eigenvalue problem. The spectrum of eigenvalues is now a semi-infinite continuum together with a finite

[5] P. D. Lax, Courant Institute of Mathematical Sciences, New York University (private communication).

[6] A. Lenard, Department of Mathematics, Indiana University (private communication).

number of discrete values; however, the constancy of the eigenvalues shown by (31) is informative only for the discrete ones (if any). The eigenfunctions of the discrete spectrum are square-integrable, whereas those of the continuous spectrum, although bounded, are not square-integrable and are thus "improper." We can associate complex reflection and transmission coefficients with these improper eigenfunctions. For an incoming plane wave at ∞ or $-\infty$, the transmission coefficient turns out to be a constant of motion and the reflection coefficient to depend on time in a very simple way. These results have been published elsewhere[7] in brief and, together with the earlier material of this section, will be elaborated in VI.

5. FURTHER CONSTANTS OF THE KdV EQUATION FROM THE ψ EQUATION

From (28), u may be expressed as a rational function in ψ and its derivatives which is homogeneous of degree zero. Thus any polynomial conserved density for the KdV equation can be transformed into a rational homogeneous conserved density of degree zero for (32). Similarly, such conserved densities for (32) are obtained from the conserved densities for (I.2) and (I.9). In addition, it is easy to verify that (32) has two rational homogeneous conserved densities of degrees other than zero, namely, ψ^2 and ψ^{-2}, which therefore obviously do not correspond to any for the KdV equation. For the periodic case, however, we can form the two associated constants, whose product

$$C \equiv \left(\int \psi^2\, dx\right)\left(\int \psi^{-2}\, dx\right) \tag{37}$$

is itself a constant and, being homogeneous of degree zero, may evidently be considered a functional of u. (This constant can be associated with either of the two distinct *nonlocal* conserved densities $\psi^2 \int \psi^{-2}\, dx$ and $\psi^{-2} \int \psi^2\, dx$.) Since there are infinitely many eigenfunctions, we have obtained a new family of infinitely many constants of motion for the KdV equation [and hence for (I.2)].

Note, however, that since any eigenfunction (except the first) has zeros, $\int \psi^{-2}\, dx$ is not properly defined. To overcome this difficulty, first assume that u is analytic. Then so is ψ,[4] and we may extend ψ by analytic continuation to some strip containing the real axis and choose a path of integration a little off the axis to avoid the zeros of ψ. Since the residue of ψ^{-2} is zero wherever $\psi = 0$ because, by (28), $\psi_{xx} = 0$ there, the value of the integral is the same whether the path goes above or below a pole. With this interpretation of the integral, the proof of its constancy remains valid. The integral turns out to equal the finite part, in the sense of Hadamard, of the integral as originally written along the real axis. Taking this now as the definition, we may relax the assumption that u be analytic as long as u is sufficiently smooth, and the integral is still a constant.

6. CONSERVED DENSITY EXPLICITLY DEPENDENT ON x AND t

It occurred to us to look for local conserved densities of the KdV equation in the form of a general function of u and a finite number of its derivatives (instead of just a polynomial) and of x and t as well. The only new one we found (besides the irrelevant universal one consisting of a general function of x and t alone) is $T = xu - \frac{1}{2}tu^2$, the complete conservation law being

$$(xu - \tfrac{1}{2}tu^2)_t + (\tfrac{1}{2}xu^2 + xu_{xx} - u_x - \tfrac{1}{3}tu^3 - tuu_{xx} + \tfrac{1}{2}tu_x^2)_x = 0. \tag{38}$$

We have shown straightforwardly that there is no other new conserved density which is a (several times differentiable) function of x, t, u, u_x, u_{xx}, u_{xxx}, and u_{xxxx}. We conjecture that there are no others at all, so that the new conserved density is unique of its type. In any case, (38) concludes the roster of all conservation laws and constants of motion known to us.

[7] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, Phys. Rev. Letters **19**, 1095 (1967).

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Properties of the S Matrix for Velocity-Dependent Potentials

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Solutions of the Schrödinger equation with a velocity-dependent potential are discussed. The analytic behavior of the S matrix as a function of the momentum is investigated, and some properties of general validity are demonstrated. The configuration of the poles of the scattering matrix in the complex-momentum plane is described in detail for the case of a spherically symmetric potential.

1. INTRODUCTION

The observation that the interaction between two nucleons is strongly repulsive at short distances, and the consequent difficulties in the treatment of such singular potentials has originated the introduction of the velocity-dependent forces in nuclear physics.[1,2] It was shown that, in the description of the two-nucleon interaction, hard-core terms in the Schrödinger equation could be replaced with advantage by velocity-dependent potentials.[3,4]

The introduction of new terms containing momentum operators changes the form of the differential equation, and a restudy of the properties of its solutions may be then necessary in such cases. In a previous paper[5] we have discussed the properties of the wavefunction, special attention being given to the consequences of the existence of singularities in the differential equation.

We may now argue whether the S matrix derived from velocity-dependent potentials presents analytic properties in the complex-momentum (or energy) plane which differ in some respect from the well-known properties of the S matrix for static potentials. The momentum-dependent terms introduced in the Schrödinger equation may cause the corresponding S matrix to present a dependence on momentum which does not fall into general forms occurring for static potentials. When analytically extended to the complex-momentum or energy plane, this S matrix may exhibit a behavior which deviates from the usual (static case) one. Thus a study of the analytic properties of the S matrix for velocity-dependent potentials seems to deserve attention. This paper is dedicated to this study. We remark that the behavior of the scattering amplitude derived from velocity-dependent potentials as a function of the complex angular momentum (i.e., its Regge poles) has been discussed recently.[6]

According to the original proposal of Razavy, Field, and Levinger,[1] we introduce in the one-particle Schrödinger equation a potential of the form

$$V(\mathbf{r}, \mathbf{p}) = V_1(\mathbf{r}) - (\lambda/2m)\mathbf{p} \cdot J(\mathbf{r})\mathbf{p}, \qquad (1.1)$$

where m is the mass of the particle, λ is a dimensionless constant, and $\mathbf{p}$ is the momentum operator. $V_1(\mathbf{r})$ and $J(\mathbf{r})$ are assumed to be real functions. This potential is invariant under the time-reversal operation, conserves parity, and is Hermitian.[7] With this potential we obtain the wave equation

$$-(\hbar^2/2m)\nabla \cdot [(1 - \lambda J)\nabla\psi] + V_1\psi = i\hbar\partial\psi/\partial t. \qquad (1.2)$$

For stationary-state wavefunctions of the form

$$\psi(\mathbf{r}, t) = \phi(\mathbf{r})e^{-iEt/\hbar} \qquad (1.3)$$

we have

$$-(\hbar^2/2m)\nabla \cdot [(1 - \lambda J)\nabla\Phi] + V_1\phi = E\phi. \qquad (1.4)$$

2. SPHERICALLY SYMMETRIC PROBLEMS; PROPERTIES OF THE S MATRIX FOR FINITE-RANGE POTENTIALS

With V_1 and J in Eq. (1.1) depending only on the radial distance r, the wave equation can be separated

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in partial waves (Ref. 5) by putting

$$\phi(\mathbf{r}) = \sum_{l,m} R_l(r) Y_{lm}(\theta, \varphi), \tag{2.1}$$

where $Y_{lm}(\theta, \varphi)$ are the spherical harmonic functions, and the radial part satisfies the differential equation

$$(1 - \lambda J)\left(R_l'' + \frac{2}{r} R_l' - \frac{l(l+1)}{r^2} R_l\right) + (d(1 - \lambda J)/dr) R_l' + (k^2 - U_1) R_l = 0, \tag{2.2}$$

with $k^2 = 2mE/\hbar^2$, $U_1 = 2mV_1/\hbar^2$. The differential equation for the reduced wavefunction $u_l(r)$, defined by

$$R_l(r) = u_l(r)/r, \tag{2.3}$$

is

$$(1 - \lambda J)\left(u_l'' - \frac{l(l+1)}{r^2} u_l\right) + \left(\frac{d(1 - \lambda J)}{dr}\right)\left(u_l' - \frac{1}{r} u_l\right) + (k^2 - U_1) u_l = 0. \tag{2.4}$$

We assume that U_1 and λJ are real functions. Combining in the usual way Eq. (2.4) and its complex conjugate, we obtain

$$d[(1 - \lambda J)(u_l' u_l^* - u_l^{*\prime} u_l)]/dr + (k^2 - k^{*2}) |u_l|^2 = 0. \tag{2.5}$$

By integrating this expression from 0 to r, with the assumption that $J(r)$ is regular at the origin, we get

$$[1 - \lambda J]_r [u_l' u_l^* - u_l^{*\prime} u_l]_r + (k^2 - k^{*2}) \int_0^r |u_l|^2 \, dr = 0. \tag{2.6}$$

If $\lambda J(r)$ and $U_1(r)$ are both finite-range potentials, that is, if

$$\lambda J(r) = 0, \quad U_1(r) = 0, \quad \text{for} \quad r \geq r_0, \tag{2.7}$$

Eq. (2.4) in the region outside the range of the potential has free-wave solutions which we call

$$f_l(k, r) = kr h_l^{(1)}(kr) \xrightarrow[r \to \infty]{} \exp\{i(kr - l\pi/2)\}, \tag{2.8a}$$

$$g_l(k, r) = kr h_l^{(2)}(kr) \xrightarrow[r \to \infty]{} \exp\{-i(kr - l\pi/2)\}, \tag{2.8b}$$

corresponding to outgoing and incoming waves, respectively. The general solution of the differential equation for $r \geq r_0$ is then

$$u_l(k, r) = A_l(k) f_l(k, r) + B_l(k) g_l(k, r), \quad r \geq r_0. \tag{2.9}$$

From the definition of the S matrix (or, more precisely, of the S_l function)

$$S_l = A_l(k)/B_l(k), \tag{2.10}$$

we then obtain

$$S_l(k) = [(g_l u_l' - g_l' u_l)/(f_l u_l' - f_l' u_l)]_{r \geq r_0}. \tag{2.11}$$

It is well known[8] that for real static potentials the S_l function has no poles in the upper part of the complex k plane except for points on the positive imaginary axis. In our case, however, the terms of the differential equation which depend on k are not real for complex values of k. Besides this, the wave equation is different from a Schrödinger equation with static potential, due to the presence of the first derivative of the wavefunction. Thus the above-mentioned property of the poles of the S_l function should not be taken as valid *a priori*. We now prove that it is in fact true in our case (in spite of the fact that it need not be true for complex potentials of arbitrary form).

Poles of $S_l(k)$ exist for values of k such that the denominator in Eq. (2.11) vanishes; that is,

$$[f_l u_l' - f_l' u_l]_{r \geq r_0} = 0. \tag{2.12}$$

Dividing Eq. (2.6) by $|u_l(r)|_r^2$, we obtain

$$[1 - \lambda J]_r [u_l'/u_l - (u_l'/u_l)^*]_r + ((k^2 - k^{*2})/|u_l|_r^2) \int_0^r |u_l|^2 \, dr = 0. \tag{2.13}$$

Bringing the condition Eq. (2.12) for the existence of a pole into Eq. (2.13), we get, for any $a > r_0$,

$$[1 - \lambda J]_a (f_l'/f_l - (f_l'/f_l)^*]_a + ((k^2 - k^{*2})/|u_l|_a^2) \int_0^a |u_l|^2 \, dr = 0. \tag{2.14}$$

Applying to the free-wave equation a technique similar to that used to obtain Eq. (2.5), we can get for the outgoing solution

$$d(f_l' f_l^* - f_l^{*\prime} f_l)/dr + (k^2 - k^{*2}) |f_l|^2 = 0. \tag{2.15}$$

We now integrate this equation from a to ∞. If we take $\operatorname{Im}(k) > 0$, we have that $f_l(\infty) = 0$, and the integral results

$$-[f_l' f_l^* - f_l^{*\prime} f_l]_a + (k^2 - k^{*2}) \int_a^\infty |f_l|^2 \, dr = 0. \tag{2.16}$$

Dividing by $|f_l|_a^2$, we get

$$-[f_l'/f_l - (f_l'/f_l)^*]_a + ((k^2 - k^{*2})/|f_l|_a^2) \int_a^\infty |f_l|^2 \, dr = 0. \tag{2.17}$$

We can now add Eqs. (2.14) and (2.17). Since, according to Eq. (2.7), we have $[1 - \lambda J]_{r=a} = 1$, there results that the condition for the existence of poles

[8] A. Martin, Nuovo Cimento **14**, 403 (1959).

with Im $(k) > 0$ is that

$$(k^2 - k^{*2})\left[\left(\int_0^a |u_l|^2\,dr\right)\Big/|u_l|_a^2 + \left(\int_a^\infty |f_l|^2\,dr\right)\Big/|f_l|_a^2\right] = 0. \quad (2.18)$$

Since the term in the brackets is positive-definite,

$$k^2 - k^{*2} \equiv 4i\,\mathrm{Re}\,(k)\,\mathrm{Im}\,(k)$$

has to be zero, and this is possible only when Re $(k) = 0$. Thus, we conclude that, in the problem here considered, poles of $S_l(k)$ can exist only in the lower k plane or in the positive imaginary axis. This is a consequence of the form assumed for the potential introduced in the Schrödinger equation. A discussion of the particular case of $l = 0$ presented in the next section will suggest a possible origin for this simple behavior.

It is known that the nonexistence of poles of the S matrix in the upper half of the complex-momentum plane has some connection with the causality principle. In relativistic theories, where there is a limit velocity for the propagation of signals, the connection of the causality principle with the analytic properties of the S matrix is well understood.[9] For nonrelativistic quantum mechanics it has been shown by van Kampen[10] that whenever it is possible to define a probability density (obeying a conservation law) in all space, it can be proved that poles of S_l can occur only on the lower part of the momentum plane and on the positive imaginary axis. Since in our problem we have the usual expression $P = \psi^*\psi$ for the probability density and a continuity equation (although the expression for the current density is different from the usual one),[5] the result derived above is in fact a consequence of van Kampen's proof.

Let us now examine the symmetry properties of $S_l(k)$. From simple inspection of Eq. (2.4) we see that its solutions satisfy

$$u_l(k, r) = C_l(k)u_l(-k, r), \quad (2.19a)$$

$$u_l(k^*, r) = D_l(k)u_l^*(k, r). \quad (2.19b)$$

On the other hand, the solutions of the free-wave equation satisfy

$$f_l(-k, r) = (-1)^l g_l(k, r), \quad g_l(-k, r) = (-1)^l f_l(k, r); \quad (2.20)$$

$$f_l(k^*, r) = g_l^*(k, r), \quad g_l(k^*, r) = f_l^*(k, r). \quad (2.21)$$

[9] N. G. van Kampen, Phys. Rev. **89**, 1072 (1952).
[10] N. G. van Kampen, Phys. Rev. **91**, 1267 (1953).

Then there results immediately that

$$S_l(-k) = [S_l(k)]^{-1} \quad (2.22)$$

and

$$S_l(k^*) = S_l^*(-k), \quad (2.23)$$

which are analogous to the properties of the S_l function for static real potentials, and say that poles and zeros of S_l are symmetric with respect to the imaginary axis in the k plane, and that if for a value k there is a pole of S_l, in $-k$ there is a zero of the same function. These results are again a direct consequence of the particular form, Eq. (1.1), assumed for the velocity-dependent potential.

3. AN "EQUIVALENT" STATIC POTENTIAL FOR THE S-WAVE CASE

We now discuss the particular case of $l = 0$, where the origin of the general properties discussed in the previous section can be better understood.

Due to the term $-\lambda(dJ(r)/dr)(dR_l(r)/dr)$, Eq. (2.2) cannot in general be written in the form of a usual Schrödinger equation. We now show how we can do it in the case of $l = 0$, with $V_1 = 0$. The s-wave radial equation is then

$$d[(1 - \lambda J)\,dR_0/dr]/dr + (2/r)(1 - \lambda J)\,dR_0/dr + k^2R_0 = 0. \quad (3.1)$$

Let us introduce a new function

$$\chi_R = (1 - \lambda J)\,dR_0/dr. \quad (3.2)$$

We obtain

$$d\chi_R/dr + 2\chi_R/r + k^2R_0 = 0. \quad (3.3)$$

Taking the derivative of this equation and eliminating dR_0/dr by using Eq. (3.2), we get

$$d^2\chi_R/dr^2 + (2/r)\,d\chi_R/dr + (k^2/(1 - \lambda J) - 2/r^2)\chi_R = 0 \quad (3.4)$$

or

$$d^2\chi_R/dr^2 + (2/r)\,d\chi_R/dr + (k^2 - U(r))\chi_R = 0, \quad (3.5)$$

where

$$U(r) = 2/r^2 - k^2\lambda J(r)/(1 - \lambda J(r)). \quad (3.6)$$

Now, Eq. (3.5) has the form of a s-wave radial Schrödinger equation with potential $(\hbar^2/2m)U(r)$, or alternatively, by considering $2/r^2 \equiv 1(1 + 1)/r^2$ as a centrifugal term, we can consider it as a p-wave radial equation with a simpler potential of assumed finite range

$$(\hbar^2/2m)k^2\mu(r) \equiv -(\hbar^2/2m)k^2\lambda J(r)/(1 - \lambda J(r)). \quad (3.7)$$

In terms of a reduced wavefunction

$$y_1(r) = r\chi_R(r), \quad (3.8)$$

Eq. (3.5) becomes

$$d^2y_1/dr^2 + k^2y_1 - k^2\mu(r)y_1 - (2/r^2)y_1 = 0. \quad (3.9)$$

Properties of the wavefunction for usual static potentials will then be valid for χ_R, and the behavior of R_0 can be obtained from Eq. (3.2) or Eq. (3.3). If R_0 is a bound-state wavefunction, that is, if it has the form $F(r)e^{-\alpha r}$ for large r [with $F(r)$ tending to a constant as $r \to \infty$], then χ_R has the form $G(r)e^{-\alpha r}$ for large r [where $G(r)$ tends to a constant; we assume that $\lambda J(r)$ becomes a constant for large r]. Thus, χ_R will also be the wavefunction for a bound state with the same binding energy.

If $R_0(r)$ has an asymptotic behavior

$$R_0(r) \approx Ae^{ikr} + Be^{-ikr}, \quad (3.10)$$

then the asymptotic behavior of χ_R is

$$\chi_R \approx Ae^{ikr} - Be^{-ikr}. \quad (3.11)$$

The scattering-matrix elements in the two problems differ only by having opposite signs, and the phase shifts are different by $\pi/2$; that is, if

$$R_0 \approx (1/r)\sin(kr + \delta_0), \quad (3.12)$$

then

$$\chi_R \approx (1/r)\sin(kr + \delta_0 + \pi/2). \quad (3.13)$$

Thus, the poles and zeros of the S-matrix element corresponding to $l = 0$ are the same in the velocity-dependent potential as in the equivalent static problem with potential $U(r)$. We must, however, notice that the energy enters $U(r)$ as a parameter, and if we study the behavior of the S matrix for complex values of the momentum, we shall have that the static potential under study is actually complex, and general results for real static potentials may not be true. Let us examine some properties of the S matrix in the case of complex potentials of the form $k^2\mu(r)$, where k can be complex and $\mu(r)$ is real and of finite range. For a wave equation of the form

$$y_l'' + k^2y_l - k^2\mu(r)y_l - [l(l+1)/r^2]y_l = 0, \quad (3.14)$$

of which Eq. (3.9) is a particular case for $l = 1$, we have free-wave solutions outside the range of the potential given by Eqs. (2.8), and the S_l function is given by Eq. (2.11) with y_l in the place of u_l. The solutions y_l of Eq. (3.14) have properties similar to those given in Eqs. (2.19), so that the symmetry properties of the S_l function expressed in Eqs. (2.22) and (2.23) again hold. These symmetry properties are a consequence of the particular form $k^2\mu(r)$ of the complex potential entering Eq. (3.14).

To prove that the S_l function does not admit poles in the upper part of the complex k plane (except for points of the imaginary axis), we can use a technique similar to that used in the previous section. We now have to multiply Eq. (3.14) by $k^{*2}y_l^*$ and the complex conjugate by k^2y_l, subtract the two expressions, and integrate from 0 to a (a being larger than the range of the potential) to obtain

$$[k^{*2}y_l^*y_l' - k^2y_ly_l^{*\prime}]_a + (k^2 - k^{*2})\int_0^a (|y_l'|^2 + (l(l+1)/r^2)|y_l|^2)\,dr = 0, \quad (3.15)$$

or, dividing by $|k^2|^2|y_l|_a^2$,

$$[y_l'/k^2y_l - (y_l'/k^2y_l)^*]_a + ((k^2 - k^{*2})/|k^2|^2|y_l|_a^2) \times \int_0^a (|y_l'|^2 + (l(l+1)/r^2)|y_l|^2)\,dr = 0. \quad (3.16)$$

From the free-wave equation we obtain for the outgoing solution, if Im $(k) > 0$,

$$-[f_l'/k^2f_l - (f_l'/k^2f_l)^*]_a + ((k^2 - k^{*2})/|k^2|^2|f_l|_a^2) \times \int_a^\infty (|f_l'|^2 + (l(l+1)/r^2)|f_l|^2)\,dr = 0. \quad (3.17)$$

The condition for existence of poles [Eq. (2.12), with y_l substituted for u_l] can be written

$$[f_l'/k^2f_l - (f_l'/k^2f_l)^*]_a + ((k^2 - k^{*2})/|k^2|^2|y_l|_a^2) \times \int_0^a (|y_l'|^2 + (l(l+1)/r^2)|y_l|^2)\,dr = 0. \quad (3.18)$$

Using Eq. (3.17), which is an identity valid for Im $(k) > 0$, we obtain

$$((k^2 - k^{*2})/|k^2|^2) \times \left[\left(\int_a^\infty (|f_l'|^2 + (l(l+1)/r^2)|f_l|^2)\,dr\right)\Big/|f_l|_a^2 + \left(\int_0^a (|y_l'|^2 + (l(l+1)/r^2)|y_l|^2)\,dr\right)\Big/|y_l|_a^2\right] = 0. \quad (3.19)$$

Since the term in the brackets is necessarily positive, it follows that poles will only exist if Re $(k) = 0$.

Thus, although we have a complex potential in the Schrödinger equation, poles do not pass to the upper plane. We can understand this in the following way. Let us think of the configuration of the poles of S_l as a function of the parameter λ which determines the strength of the potential. As the poles of S_l are given by the zeros of an analytic function [which is the denominator of Eq. (2.11)], they move continuously in the k plane as λ varies, without new poles being created or existing poles being destroyed. It is known[11] that for very weak potentials the poles are situated far down in the k plane, and for all of them $\mathrm{Im}(k) \to -\infty$ as $\lambda \to 0$. As λ increases, the poles may move up to the k plane and try to cross the real axis. But as k approaches the real axis, the potential becomes real, and it is well known that the S_l function for a real potential cannot present a pole on the real k axis. Thus, it becomes forbidden for any pole to pass to the upper half-plane by crossing the real axis. An alternative which remains is that of a pole climbing up the imaginary axis, crossing the real axis at the origin (or just reaching the origin), and then passing to the complex plane. But now we can show that once the pole is in the positive imaginary axis, it does not leave it. For imaginary values of k the potential is real (since it depends only on k^2); if λ changes by a small amount, there can always exist a change of the position of the pole along the imaginary axis such that the new position of the pole will correspond to a pole for a real potential with the appropriate strength. If the pole can move along the imaginary axis as λ changes, it cannot move in any other direction since a pole never splits itself into several poles.

4. VELOCITY-DEPENDENT SQUARE WELL OR BARRIER; POLES OF S_l

We shall study in some detail the problem of a velocity-dependent spherically symmetric constant potential; that is, in Eq. (1.1) we shall put $V_1 = 0$ and

$$J(\mathbf{r}) = 1, \quad r < b, \qquad = 0, \quad r > b. \tag{4.1}$$

We shall be mainly concerned with the description of the distribution of the poles of S_l function in the momentum plane, as a function of the strength λ of the potential. The analytic properties of the S matrix for such potential in the complex angular momentum plane have been studied by Weigel.[6] We here only consider real integer values of the angular momentum.

The radial wave equation for this potential becomes

$$(1 - \lambda J(r))(R_l'' + (2/r)R_l') \times (k^2 - (1 - \lambda J(r))l(l+1)/r^2)R_l = -\lambda R_l'\delta(r - b). \tag{4.2}$$

The internal solution which is regular at the origin and the general external solution are given by

$$R_{l\mathrm{I}}(r) = B_l j_l(k'r), \qquad r < b, \tag{4.3a}$$

$$R_{l\mathrm{II}}(r) = A_l[j_l(kr) - \tan\delta_l n_l(kr)], \quad r > b, \tag{4.3b}$$

where

$$k' = k/[1 - \lambda]^{\frac{1}{2}} \tag{4.4}$$

is the wavenumber inside the range of the potential.

The continuity conditions from the solution of the differential equation[5] impose that

$$R_{l\mathrm{I}}(b) = R_{l\mathrm{II}}(b), \tag{4.5a}$$

$$(1 - \lambda)R_{l\mathrm{I}}'(b) = R_{l\mathrm{II}}'(b). \tag{4.5b}$$

We obtain for the phase shift

$$\tan\delta_l = \frac{kj_l'(kb)j_l(k'b) - (1-\lambda)k'j_l'(k'b)j_l(kb)}{kn_l'(kb)j_l(k'b) - (1-\lambda)k'j_l'(k'b)n_l(kb)} \tag{4.6}$$

and for the S_l function

$$S_l(kb) = -\frac{kh_l^{(2)\prime}(kb)j_l(k'b) - (1-\lambda)k'j_l'(k'b)h_l^{(2)}(kb)}{kh_l^{(1)\prime}(kb)j_l(k'b) - (1-\lambda)k'j_l'(k'b)h_l^{(1)}(kb)}. \tag{4.7}$$

If we write the spherical Hankel functions in the form

$$h_l^{(1)}(\rho) = -iM_l(\rho)e^{i\rho}/\rho, \tag{4.8a}$$

$$h_l^{(2)}(\rho) = iN_l(\rho)e^{-i\rho}/\rho, \tag{4.8b}$$

where $M_l(\rho)$ and $N_l(\rho)$ are polynomials of degree l in $1/\rho$, we can then write the S_l function as

$$S_l(kb) = e^{-2ikb}F_l(kb), \tag{4.9}$$

where

$$F_l(kb) = \frac{kj_l(k'b)[lN_{l-1}(kb) - (l+1)N_{l+1}(kb)] - (1-\lambda)k'j_l'(k'b)N_l(kb)}{kj_l(k'b)[lM_{l-1}(kb) - (l+1)M_{l+1}(kb)] - (1-\lambda)k'j_l'(k'b)M_l(kb)} \tag{4.10}$$

is a meromorphic function with isolated poles determined by the zeros of the denominator. The factor e^{-2ikb} in $S_l(kb)$ shows that it presents essential singularities in the upper plane for $\mathrm{Im}(k) \to \infty$. This behavior is the same as that observed in the cases of static potentials. From well-known properties of the spherical Bessel functions [j_l and n_l are functions of parity $(-1)^l$ and $(-1)^{l+1}$, respectively, and are real functions when their arguments are real], we can easily verify the validity of the symmetry properties of $S_l(kb)$ given in Eqs. (2.23) and (2.24). Since these

[11] J. Humblet, Mem. Soc. Roy. Sci. Liège 12, No 4, 70 (1952).

properties say that the poles of S_l are symmetric with respect to the imaginary axis, we need only describe the configuration of poles in one half-plane, let us say Re $(k) \geq 0$.

The equation which determines the poles of $S_l(kb)$ is obtained by putting the denominator in Eq. (4.7) equal to zero. This condition can be written

$$kb[[1 - \lambda]^{\frac{1}{2}} j_{l+1}(k'b)/j_l(k'b) - h_{l+1}^{(1)}(kb)/h_l^{(1)}(kb)] + \lambda l = 0. \quad (4.11)$$

We shall next discuss the behavior of the roots of this equation.

A. The Limit of Weak Potentials

If in Eq. (4.11) we put $\lambda = 0$, we get

$$j_{l+1}(k\ )/j_l(kb) - h_{l+1}^{(1)}(kb)/h_l^{(1)}(kb) \equiv i/[(kb)^2 j_l(kb) h_l^{(1)}(kb)] = 0,$$

which has no solution for any finite value of kb. Thus, as $\lambda \to 0$ the poles of $S_l(kb)$ must tend to infinity. Taking asymptotic forms of the Bessel and Hankel functions valid for $|kb| \to \infty$ and $|k'b| \to \infty$, we obtain for the equation determining the poles as $\lambda \to 0$

$$\exp\{-2i(k'b - l\pi/2)\} \approx (\lambda/4)[1 + \lambda/4 + i(l^2 + l + 2)/kb]. \quad (4.12)$$

Let us write

$$k = x + iy, \quad (4.13)$$

where x is the real part and y the imaginary part of k. Analogously we write

$$k' = x' + iy'. \quad (4.14)$$

Taking the modulus of Eq. (4.12) we obtain

$$\exp\{2y'b\} = (|\lambda|/4)[(1 + \lambda/4 + yb(l^2 + l + 2)/(x^2b^2 + y^2b^2))^2 + (xb(l^2 + l + 2)/(x^2b^2 + y^2b^2))^2]^{\frac{1}{2}}. \quad (4.15)$$

Since the right-hand side of this equation goes to zero with λ, we must have $y'b \to -\infty$ (and then also $yb \to -\infty$) as $\lambda \to 0$ so that the left-hand side also goes to zero. We thus conclude that the poles move downwards in the complex k plane as the potential becomes weaker. The values of xb at the poles may remain finite and, taking the main terms of Eq. (4.15), we obtain

$$\exp\{2yb/[1 - \lambda]^{\frac{1}{2}}\} \simeq (|\lambda|/4)[1 + \lambda/4 + yb(l^2 + l + 2)/(x^2b^2 + y^2b^2)], \quad (4.16)$$

and then, to the lowest orders in λ,

$$yb \simeq (\tfrac{1}{2}) \ln |\lambda| - \ln 2. \quad (4.17)$$

Thus, all poles tend to acquire the same value of yb as $|\lambda| \to 0$: they move downwards in the k plane in such a way that their distance to the real axis tend to be the same. From Eq. (4.16) it can be seen that the larger the value of xb for a given pole, the more delayed it is in moving downwards as $|\lambda| \to 0$.

Separating real and imaginary parts in Eq. (4.12), we obtain

$$(-1)^l \exp\{2y'b\} \cos(2x'b) = (\lambda/4)[1 + \lambda/4 + yb(l^2 + l + 2)/(x^2b^2 + y^2b^2)], \quad (4.18a)$$

$$-(-1)^l \exp\{2y'b\} \sin(2x'b) = (\lambda/4)xb(l^2 + l + 2)/(x^2b^2 + y^2b^2). \quad (4.18b)$$

Dividing one of these equations by the other, we get

$$\tan(2x'b) = -(xb(l^2 + l + 2)/(x^2b^2 + y^2b^2))/(1 + \lambda/4 + yb(l^2 + l + 2)/(x^2b^2 + y^2b^2)), \quad (4.19)$$

from which it is easily seen that, as $\lambda \to 0$,

$$\tan(2x'b) \to 0 - \epsilon xb,$$

where ϵ is a positive infinitesimal quantity. Thus, the poles move towards asymptote lines defined by

$$xb = n\pi/2, \quad n \text{ integer}, \quad (4.20)$$

in such a way that they approach these lines from the left [we are discussing only the half-plane Re $(k) > 0$; the situation is symmetric for poles with Re $(k) < 0$]. This behavior is different from that observed with weak static potentials, where poles approach the asymptote lines $xb = n\pi/2$ always from the right.[12,13]

If we take Eq. (4.20) into Eq. (4.18a), we obtain that the sign of the left-hand side of Eq. (4.18a) is $(-1)^{l+n}$, while the sign of its right-hand side is that of λ. Thus, as $\lambda \to 0^+$, the poles tend to asymptote lines $xb = n\pi/2$ such that $l + n$ is even; in other words, the values of xb tend to even multiples of $\pi/2$ if l is even and to odd multiples if l is odd. This behavior is observed in attractive static potentials whose strength tends to zero [note the minus sign that precedes our potential in Eq. (1.1), so that the sign of the static potential is the same as that of $-\lambda/2m$].

On the other hand, as $\lambda \to 0^-$, the asymptotes are lines $xb = n\pi/2$ with n odd when l is even, and with n even when l is odd, just as in the case of static repulsive potentials.

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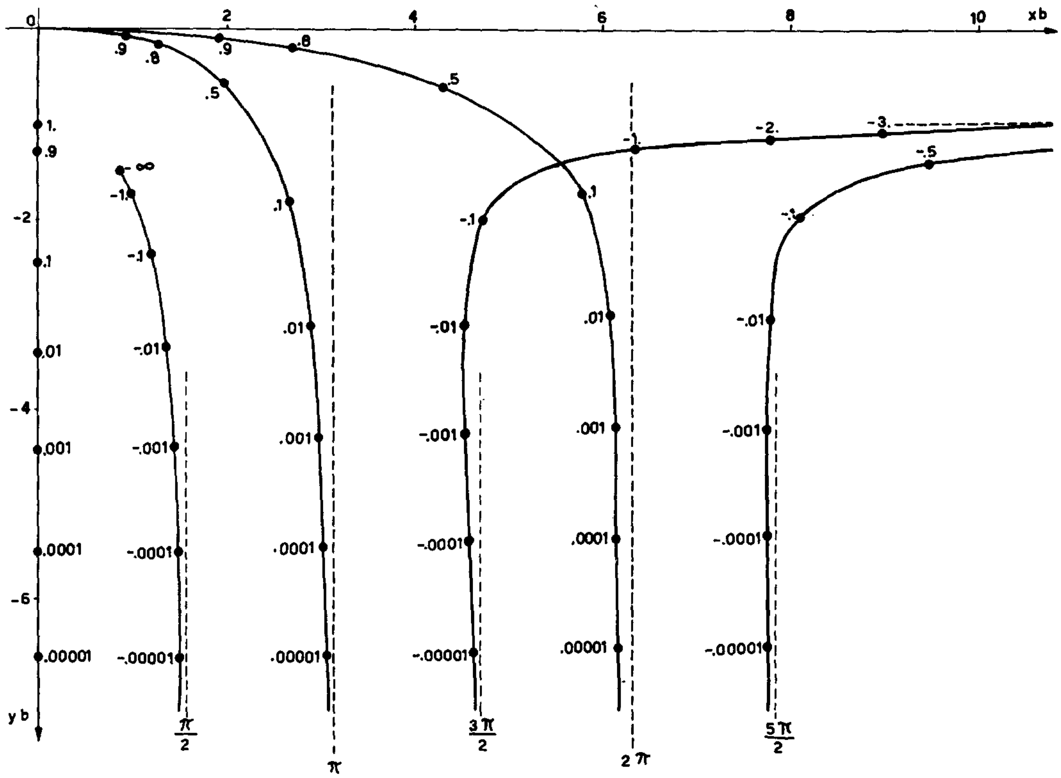


FIG. 1. Trajectories of poles of S_l for $l = 0$ shown in the kb plane, for λ varying from $-\infty$ to $+1$. Values of λ are indicated on the curves. For $\lambda \to 0^-$ the poles go to infinity approaching asymptotically the lines $xb = n\pi/2$ with n odd. As $\lambda \to -\infty$, all poles, except the so-called special pole, move towards infinite values of xb, approaching asymptotically the horizontal line $yb = -1$. The special pole remains in the finite kb plane as $\lambda \to -\infty$ reaching the point $(3)^{\frac{1}{2}}/2 - (\frac{3}{2})i$ in this limit. As $\lambda \to 0^+$ the poles move towards the asymptote lines $xb = n\pi/2$ with n even integer. As λ increases to 1 all poles, except the special one, move towards the origin. For λ varying from 0^+ to 1 the two symmetric special poles move as a double pole upwards along the imaginary axis, reaching the point $kb = -i$ as λ reaches 1.

This behavior of the poles as $|\lambda| \to 0$ can be seen in Figs. 1 and 2, where trajectories of the s- and p-wave poles in the kb plane are shown..

B. The Limit $\lambda \to -\infty$

We have just seen that as $\lambda \to 0^-$ the poles of S_l tend to asymptote lines $xb = n\pi/2$ with n being odd for even l and being even for odd l. We notice that in the limits $|\lambda| \to 0$ the values k' and k tend to be the same. In the Eq. (4.11) defining the position of the poles, λ enters into k' through the relation $k' = k/[1 - \lambda]^{\frac{1}{2}}$. We thus do not expect any very peculiar behavior of the poles as λ varies from 0 to $-\infty$. Let us then study directly what happens in the limit of λ being negative and very large.

We first examine the existence of solutions of Eq. (4.11) with $\lambda \to -\infty$, with the assumption that $|kb| \to \infty$. This may lead to solutions with finite values of $|k'b|$. Using asymptotic formulas for the Hankel functions, we obtain that Eq. (4.11) becomes

$$kb([1 - \lambda]^{\frac{1}{2}} j_{l+1}(k'b)/j_l(k'b) + i) + \lambda l = 0. \quad (4.21)$$

Dividing by λ and taking the limit $\lambda \to -\infty$, we get

$$-k'b j_{l+1}(k'b)/j_l(k'b) + l = 0, \quad (4.22)$$

which has solutions given by

$$y'b = 0, \quad (4.23a)$$

$$x'b j_{l+1}(x'b) - l j_l(x'b) = 0. \quad (4.23b)$$

Equation (4.23b) has an infinite number of solutions. (We exclude from these the solution $x'b = 0$, since this case demands a special treatment which will be made later.) For instance, in the case of $l = 0$, these solutions are the values of $x'b$ which equal the roots of the Bessel function j_1. We then have that, as λ varies from 0^- to $-\infty$, the poles move in the $k'b$ plane from the lower regions of the plane towards the real axis (as λ varies from 0^- to $-\infty$, the values of $y'b$ at the poles vary from $-\infty$ to 0). The displacement of the poles in the $k'b$ plane for the cases $l = 0$ and $l = 1$ is shown in Figs. 3 and 4.

These finite values of $x'b$ when $\lambda \to -\infty$ correspond to infinite values of $xb = x'b[1 - \lambda]^{\frac{1}{2}}$. In the Appendix it is shown that the corresponding values of yb tend to

$$yb = -[1 - l(l + 1)/(x_n'b)^2]^{-1}, \quad (4.24)$$

where $x_n'b$ are the roots of Eq. (4.23b). Thus, in the k plane, as λ varies from 0^- to $-\infty$ the poles move

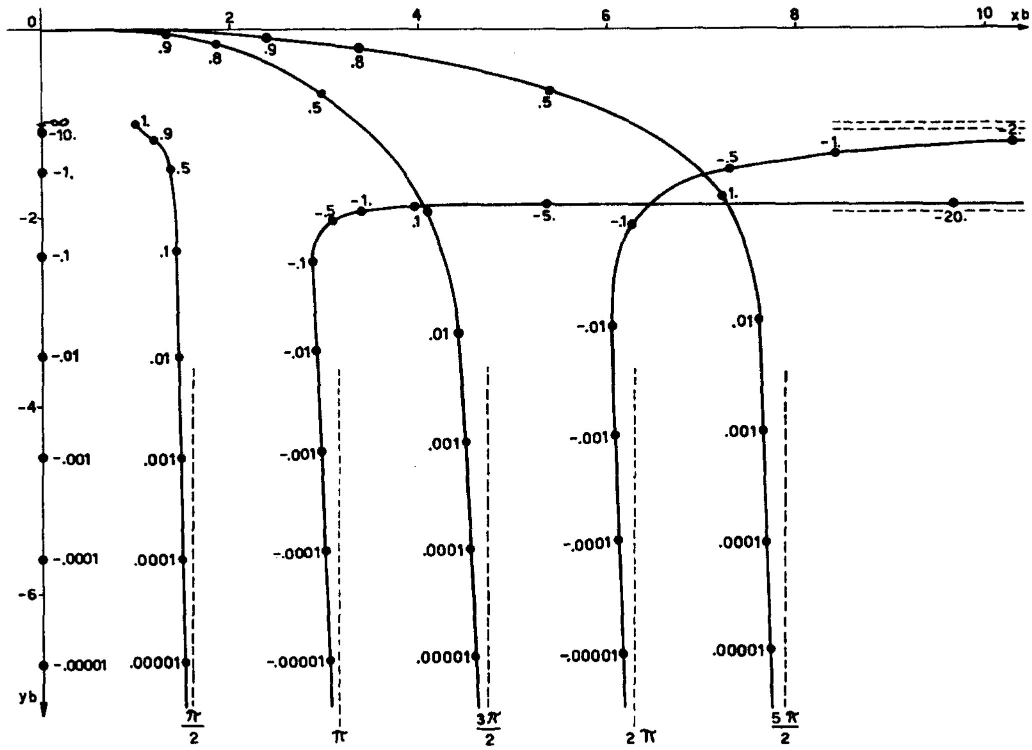


FIG. 2. Trajectories of the poles of S_l for $l = 1$, shown in the kb plane for λ varying from $-\infty$ to $+1$. The numbers on the curves indicate the values of λ. For $\lambda \to 0^-$ and for $\lambda \to 0^+$ the poles tend asymptotically to the vertical lines $xb = n\pi/2$, with n even and odd integers, respectively. As λ increases to the value 1, a special pole moves towards the point $kb = 1 - i$, and all other poles move to the origin. As λ varies from 0^- to $-\infty$ the special pole and its symmetric form a double pole which moves upwards along the negative imaginary axis, tending to the point $kb = -i$. As $\lambda \to -\infty$ all other poles move toward horizontal asymptote lines defined by Eq. (4.24) of the text.

from the lower regions of the plane towards finite values of yb and infinite values of xb. That is, the trajectories described by the poles are asymptotic to vertical lines when λ is very small negative and asymptotic to horizontal lines, given by Eq. (4.24), when $\lambda \to -\infty$. This behavior is illustrated for s and p wave poles in Figs. 1 and 2.

The asymptote lines, Eq. (4.24), all coincide with the same line $yb = -1$ when $l = 0$; for $l \geq 1$ the asymptotes are all different, all being below the line $yb = -1$, which is a kind of "accumulation asymptote line."

Let us now examine the possibility of the existence of solutions of Eq. (4.11) with $\lambda \to -\infty$, with the assumption that $|kb|$ remains finite. We then have that $|k'b| \to 0$ and using the well-known formulas for the Bessel functions for small arguments, Eq. (4.11) becomes

$$kb[kb/(2l+3) - h_{l+1}^{(1)}(kb)/h_l^{(1)}(kb)] + \lambda l = 0. \quad (4.25)$$

For $l = 0$, this equation has solutions given by

$$kb = \pm(3)^{\frac{1}{2}}/2 - (\tfrac{3}{2})i,$$

symmetrically placed in the k plane. For $l \geq 1$, the solutions of Eq. (4.25) with $\lambda \to -\infty$ are given by the roots of

$$h_l^{(1)}(kb) = 0. \quad (4.26)$$

This last equation has l solutions. [For instance, for $l = 1$, there is only one solution $kb = -i$; for $l = 2$, the roots are $kb = \pm\frac{3}{2} - ((2)^{\frac{1}{2}}/2))i$.]

We then have, in the limit $\lambda \to -\infty$, beside the poles which go to infinity in the kb plane in search of the asymptote lines given by Eq. (4.24), l poles (counting poles in both sides of the complex plane) which tend to points in the finite kb plane; in the special case $l = 0$ these poles are in number of two. In the $k'b$ plane, they tend to the origin as $\lambda \to -\infty$. We shall call them "special poles." They will be discussed in more detail in Secs. 4D and 4E.

C. The Limit $\lambda \to 1$

In the limit $\lambda \to 1$, for $k'b$ finite kb goes to zero, and if kb is kept finite, nonzero, $k'b$ increases without limit. Let us first consider the possibility that as $\lambda \to 1$, Eq. (4.11) presents solutions with finite values of $|k'b|$. Since then $|kb|$ becomes infinitesimal, we can write the equation determining the poles as

$$kb[1-\lambda]^{\frac{1}{2}} j_{l+1}(k'b) + (-(2l+1) + \lambda l) j_l(k'b) = 0, \quad (4.27)$$

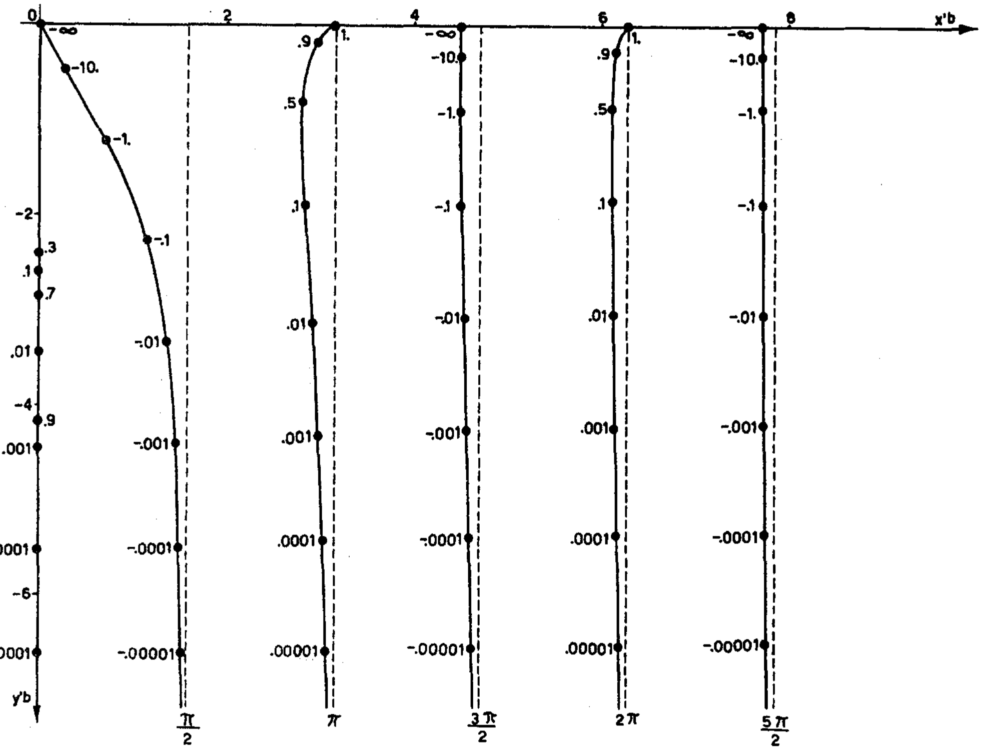


FIG. 3. Poles of S_l for $l = 0$, shown in the $k'b$ plane with λ varying from $-\infty$ to 1. Some values of λ are indicated on the curves. As $\lambda \to 0^+$ and $\lambda \to 0^-$, the poles tend asymptotically to the vertical lines $x'b = n\pi/2$, with n even and odd, respectively. As λ increases from 0^+, the poles move upwards, reaching the roots of $j_0(k'b)$ as λ reaches the value 1. As λ varies from 0^- to $-\infty$, the poles also move upwards, tending to the roots of $j_1(k'b)$ as $\lambda \to -\infty$. There is a special (double) pole which moves upwards along the negative imaginary axis as λ varies from 0^+ to about 0.3, and moves downwards along the same axis as λ is further increased, tending to infinity as λ approaches 1.

or, with $\lambda = 1$,

$$j_l(k'b) = 0. \tag{4.28}$$

This equation has an infinite number of roots given by

$$y'b = 0, \tag{4.29a}$$

$$j_l(x'b) = 0. \tag{4.29b}$$

Thus, in the $k'b$ plane, as $\lambda \to 1$, the poles tend to points on the real axis. In the kb plane all these infinitely many poles move towards the origin and get there when λ becomes equal to 1. This behavior is shown in Figs. 1, 2, 3, 4, for s- and p-wave poles. Let us now examine what happens when λ increases above 1.

For $\lambda > 1$ we have $[1 - \lambda]^{\frac{1}{2}} = \pm i[\lambda - 1]^{\frac{1}{2}}$, so that $k'b = \pm(yb - ixb)/[\lambda - 1]^{\frac{1}{2}}$ and keeping the definitions $x'b = \mathrm{Re}\,(k'b)$ and $y'b = \mathrm{Im}\,(k'b)$, we have that now

$$x'b = \pm yb/[\lambda - 1]^{\frac{1}{2}}, \tag{4.30a}$$

$$y'b = \mp xb/[\lambda - 1]^{\frac{1}{2}}. \tag{4.30b}$$

Thus, for λ larger than 1, the planes k and k' (which for $\lambda < 1$ only differ by a scale factor depending on λ) are related to each other by a rotation of $\pi/2$. The sense of the rotation is defined by the choice of $+$ or $-$ sign in Eqs. (4.30). For definiteness we shall choose the upper signs in these equations. We know that in the k plane S_l is symmetric with respect to the imaginary axis. [This can be observed directly in Eq. (4.11) by taking its complex conjugate and using properties of the Hankel functions.] For $\lambda < 1$ the same symmetry exists in the k' plane. For $\lambda > 1$, however, the relation between the two variables is given by Eqs. (4.30) and the left–right symmetry in the k plane will correspond to an up–down symmetry with respect to the real axis of the k' plane. Thus, the poles of S_l for $\lambda > 1$ will be distributed symmetrically with respect to the real axis of the k' plane.

We have seen that as $\lambda \to 1 - 0$, the poles move to the points of the real axis of the k' plane given by Eqs. (4.29). As λ increases above 1, the poles move along the x' axis; correspondingly, in the k plane the poles will be moving over the imaginary axis. Poles can only leave the x' axis (or the y axis) when they meet in pairs, so that they can pass symmetrically) to the complex plane. Since for $\lambda = 1$ the poles are separated by finite distances in the x' axis [as given by the roots of $j_l(x'b)$], they cannot move immediately into the complex plane; λ would have

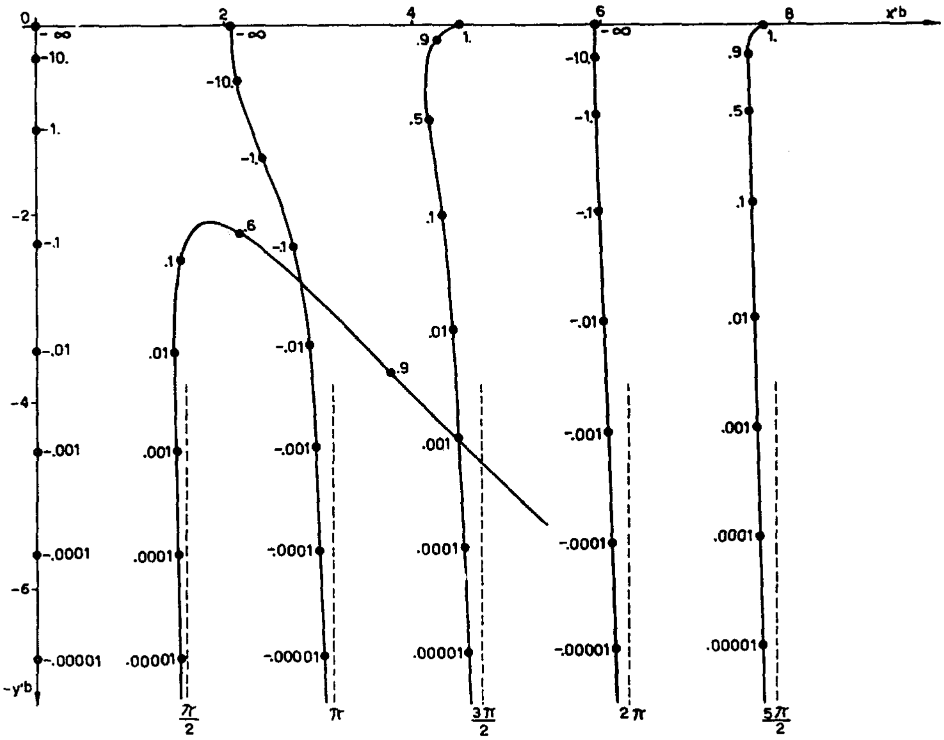


FIG. 4. Poles of S_l for $l = 1$, for λ varying from $-\infty$ to 1, shown in the $k'b$ plane. Values of λ are indicated on the curves described by the poles. As $\lambda \to 0^+$ and 0^- the poles approach asymptotically the vertical lines $x'b = n\pi/2$, with n odd and even, respectively. With $\lambda \to -\infty$ the poles tend to the roots of $k'bj_2(k'b) - j_1(k'b) = 0$. For $\lambda = 1$, an infinite number of poles are located in the points determined by $j_1(k'b) = 0$. There is a special pole which moves upwards as λ varies from 0^+ to about $+0.4$, and then starts moving downwards, returning to infinity as λ increases to 1.

to vary for finite amounts so that two poles are able to meet. We thus conclude that, as λ increases from 1, all poles in the positive (or negative, depending on the sign chosen for the rotation) x' axis will correspond to poles moving in the positive imaginary axis of the k plane. Thus, an infinite number of bound states appear suddenly as λ passes the value 1. In the case of the $l = 0$ wave this behavior is easily understood in terms of the "equivalent" static potential, by observing that for λ slightly larger than 1, $U(r)$ as given by Eq. (3.6) is strongly attractive for negative values of the energy.

Figures 5(a) and 5(b) show the displacement of s- and p-wave poles along the imaginary axis of the kb plane as λ varies above 1. It is seen that as λ increases, the poles continuously move away from the origin. This means that for λ larger than 1 the binding energy of the bound states increases with λ. Figures 6(a) and 6(b) show the displacement of the poles along the real axis in the $k'b$ plane for $\lambda > 1$.

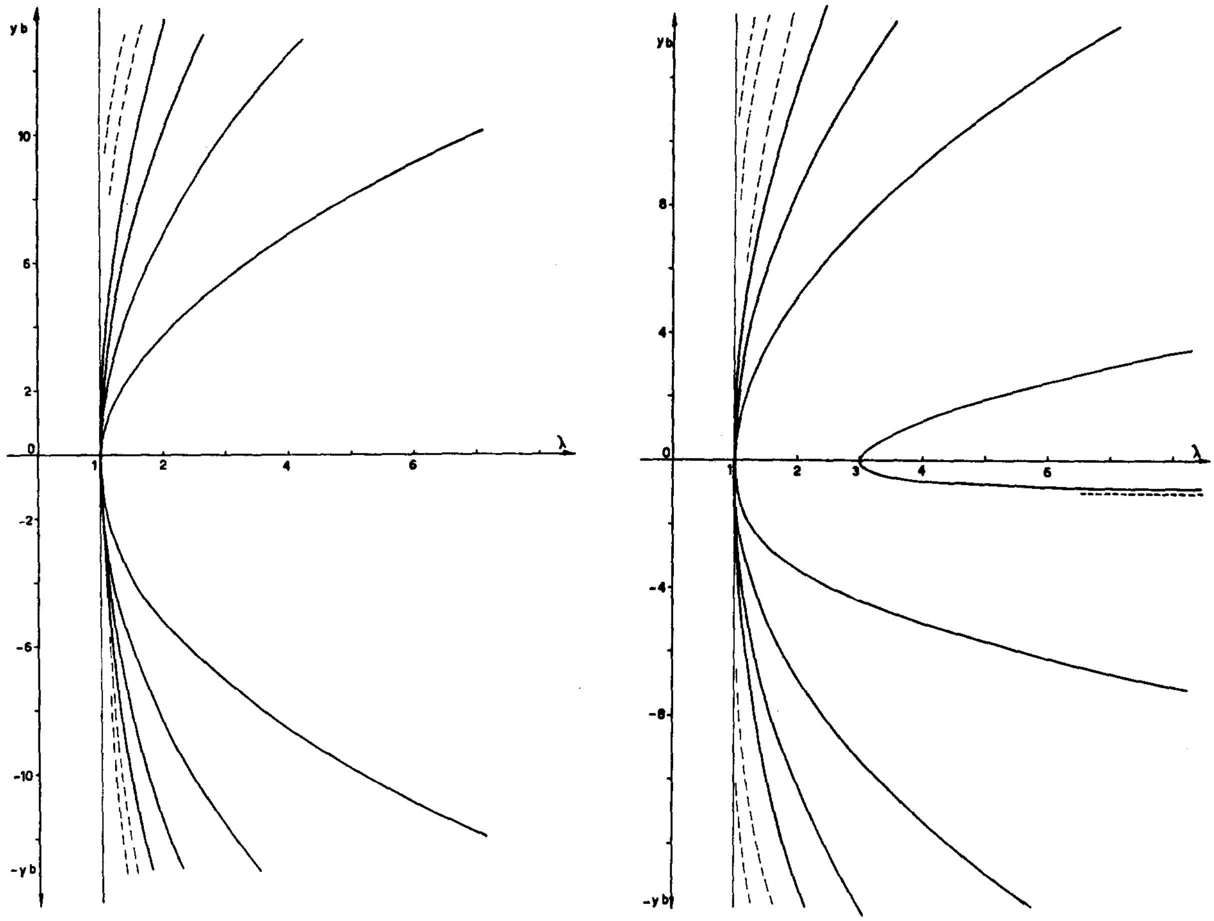
D. Special Poles

We have thus studied the behavior of the poles in the limit $\lambda \to 1$ with the assumption that $k'b$ remains finite (and then the values of kb at the poles approach zero). We now have to consider the possibility of $k'b$ increasing to infinity, so that it cannot be said *a priori* whether or not kb is infinitesimal. The quotient $j_{l+1}(k'b)/j_l(k'b)$ in Eq. (4.11) will be finite when $|k'b| \to \infty$ except in points for which $j_l(k'b) = 0$. But we have already studied the solution given by this condition. In points such that $j_l(k'b) \neq 0$, Eq. (4.11) becomes, as $\lambda \to 1$,

$$kbh_{l+1}^{(1)}(kb)/h_l^{(1)}(kb) - l = 0. \qquad (4.31)$$

This equation has $l + 1$ solutions, which we shall denote $\bar{k}b$, all in the finite kb plane. The poles corresponding to these solutions will be called "special poles." They are the only poles which do not tend to the origin of the kb plane as $\lambda \to 1$. For $l = 0$ the only solution of Eq. (4.31) is at $\bar{k}b = -i$; for $l = 1$ we have $\bar{k}b = \pm 1 - i$. In fact, among these special poles, those which are on the negative imaginary axis are double poles, and for larger values of λ they may leave the imaginary axis and appear as two symmetric complex poles. Thus we can say more properly that we have $l + 1$ special poles for l odd and $l + 2$ special poles for l even.

Let us examine the way in which these points $\bar{k}b$ are approached as λ varies. For points satisfying Eq.



(a) (b)

FIG. 5. (a) Displacement of the imaginary poles of S_l for $l = 0, \lambda > 1$. All poles, except the two so-called special poles, come from the complex plane, enter the origin for $\lambda = 1$, and move along the imaginary axis as λ is further increased. The special poles cross the imaginary axis at the point $\bar{k}b = -i$ an infinite number of times when λ varies between 1 and 1.05. (b) Displacement of the imaginary poles of S_l for $l = 1, \lambda > 1$. All poles except the two "special poles" come from the complex plane, enter the origin for $\lambda = 1$, and move to infinity along the imaginary axis as λ increases. The special poles enter the origin for $\lambda = 3$; one of them moves to infinity along the positive imaginary axis, and the other moves along the negative axis tending to the point $\bar{k}b = -i$ as λ increases to $+\infty$.

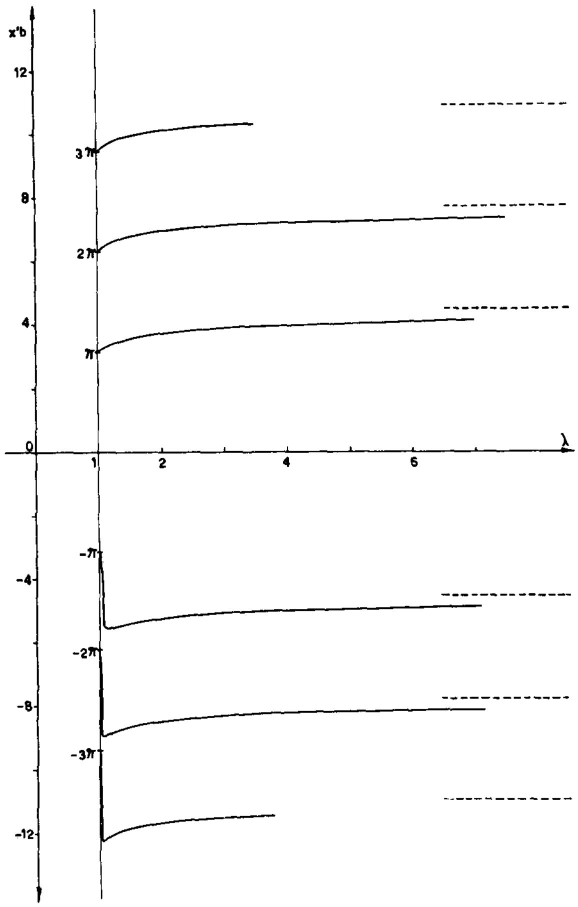
(4.31), we have that poles, as determined by Eq. (4.11), will exist for values of λ which satisfy

$$(1 - \lambda)[\bar{k}'b j_{l+1}(\bar{k}'b)/j_l(\bar{k}'b) - l] = 0. \qquad (4.32)$$

For $\lambda = 1$, Eq. (4.32) is obviously satisfied, and Eq. (4.31) defines the position of the special poles. For $\lambda \neq 1$, we can think of Eq. (4.32) as an equation determining which are the values of λ for which the solutions $\bar{k}b$ are actual locations of poles. For the points $\bar{k}b$ which are pure imaginary (there exists one such solution whenever l is even), $\bar{k}'b = \bar{k}b/[1 - \lambda]^{\frac{1}{2}}$ will be real for $\lambda > 1$. In these cases the square brackets in Eq. (4.32) will be zero for an enumerable infinity of values of λ. This comes from the fact that the Bessel function of real arguments are oscillating functions. Since the argument $k'b$ varies rapidly as λ approaches 1, we shall have poles passing very many times through the points $\bar{k}b$. Thus these points $\bar{k}b$, if on the imaginary axis, are accumulation points of poles as λ approaches 1 from values above 1. For λ less than 1, the arguments of the Bessel functions will not be real, and the term inside the brackets will not have zeros, in general, since Eq. (4.32) will have both real and imaginary parts. In the case $l = 0$, Eq. (4.32) becomes $j_1(\bar{k}'b) = 0$, and since $\bar{k}b = -i$ in this case, a pole will pass this point whenever λ is such that

$$j_1(1/[\lambda - 1]^{\frac{1}{2}}) = 0, \qquad (4.33)$$

that is, for $\lambda = 1.0495$, 1.0167, 1.0084, 1.0051, 1.0033, and thus successively, the roots converging to 1. Between two successive roots, the pole will make a small round trip around the complex plane, i.e., a pair of poles will describe paths symmetrical with respect to the imaginary axis. This same behavior



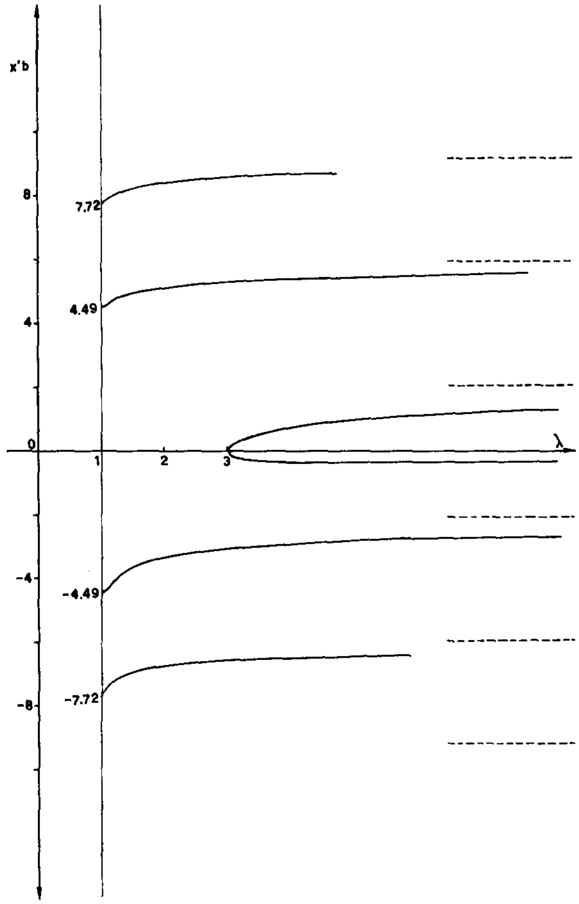


FIG. 6. (a) Displacement of the poles of S_l for $l = 0$ along the real axis of the $k'b$ plane for $\lambda > 1$. As λ varies from 1 to ∞, the poles move from points corresponding to the roots of $j_0(x'b)$ to the points determined by the roots of $j_1(x'b)$. (b) Displacement of the poles of S_l for $l = 1$ along the real axis of the $k'b$ plane for $\lambda > 1$. As λ varies from 1 to ∞, the poles move from the points corresponding to the roots of $j_1(x'b)$ to those determined by the roots of $(x'b)j_2(x'b) - j_1(x'b) \equiv f(x'b)$. The two special poles reach the origin of the $k'b$ plane for $\lambda = 3$. One of them moves towards the first nonzero root of $f(x'b)$ as λ increases. The other special pole makes a small trip along the real axis, returning to the origin as $\lambda \to \infty$.

occurs for all imaginary special poles presented by the even l waves. Figure 7 illustrates the behavior of the special poles in the case $l = 0$.

For points $\bar{k}b$ which are complex, the arguments of the Bessel functions will again be complex for both $\lambda > 1$ and $\lambda < 1$, and Eq. (4.32) will have both real and imaginary parts, which will not likely both become zero for the same real value of λ.

We have seen that all poles (there is an infinity of them), except those few called "special" poles, enter the origin of the kb plane for $\lambda = 1$. We may now ask whether there are poles which enter the origin for other values of λ. In Eq. (4.11), making $kb \to 0$ and assuming $\lambda \neq 1$ (which implies that $k'b \to 0$), we obtain, by using the well-known behavior of Bessel and Hankel functions for small arguments,

$$-(2l + 1) + \lambda l = 0. \tag{4.34}$$

For $l = 0$ there are no values of λ which satisfy this equation. For $l \geq 1$, we see that poles enter the origin for $\lambda = 2 + 1/l$. Substituting this expression for λ back into Eq. (4.11), we can obtain information on the nature of these special poles as they reach the origin. By using the expressions for the Bessel and Hankel functions with small arguments, we can easily verify that they are actually double poles for every $l \geq 1$. That is, two of the special poles enter the origin for $\lambda = 2 + 1/l$. These two poles come along paths which are symmetric about the imaginary axis of the kb plane. For $l = 1$, the special poles are in number of two, and they both enter the origin for $\lambda = 3$. This can be seen in Fig. 8, where the paths described by the special poles in the case $l = 1$ are shown. For $l > 1$, we shall have that some poles will never enter the origin (and thus will never become bound-state poles). This is due to the fact that there exist $l + 1$ special

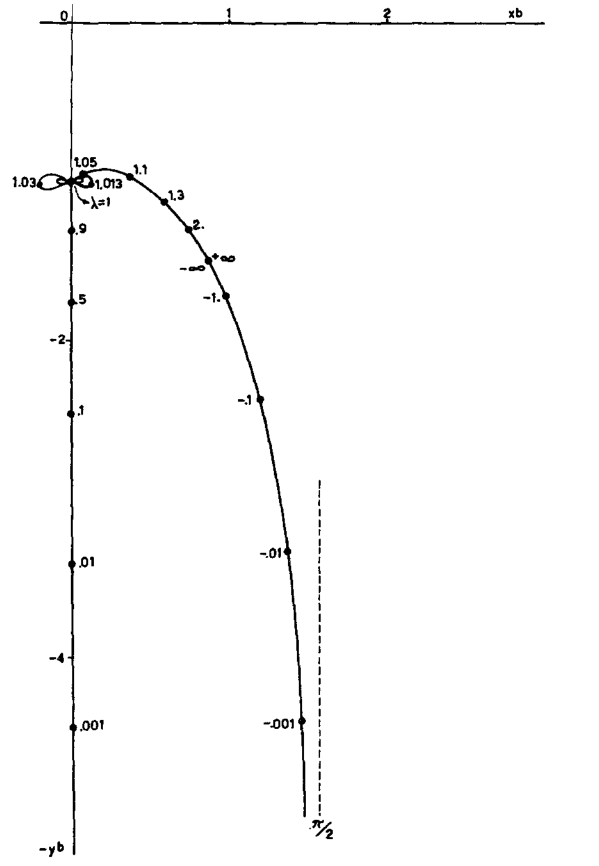


FIG. 7. Trajectories of the special poles for $l = 0$, shown in the kb plane. Some values of λ are indicated on the corresponding points of curves described by the poles. There are two special poles, symmetric with respect to the imaginary axis. For λ varying from $-\infty$ to 0^-, the poles move from $kb = \pm(3)^{\frac{1}{2}}/2 - (\frac{1}{2})i$ to infinity approaching asymptotically the lines $xb = \pm\pi/2$. From $\lambda = 0^+$ to $\lambda = 1$, a double pole moves up the negative imaginary axis, coming from infinity and approaching the point $kb = -i$ as λ approaches 1. As λ increases further the poles leave the imaginary axis, describing an infinite number of loops of increasing size, all passing the point $kb = -i$. For $\lambda \to \infty$, the poles return to the location they had for $\lambda \to -\infty$.

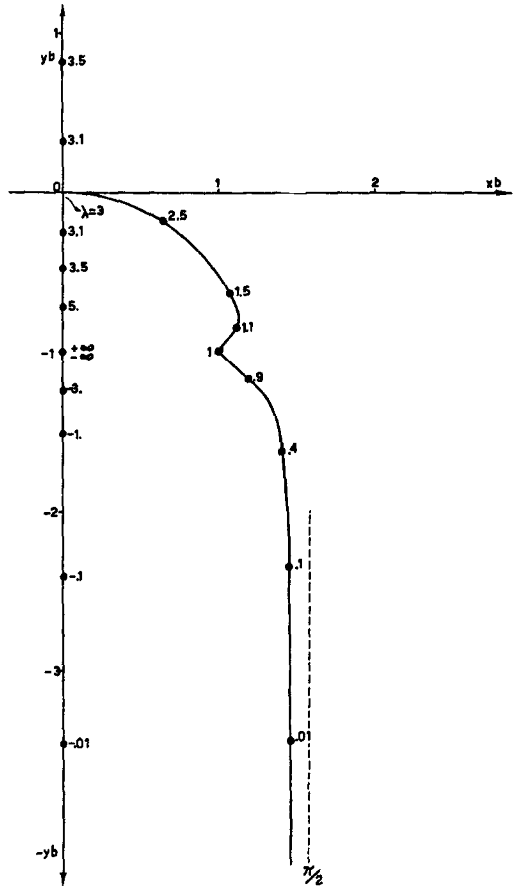


FIG. 8. Special poles for $l = 1$. The numbers next to the curves indicate the values of λ. There are two special poles, symmetric with respect to the imaginary axis. For λ varying from $-\infty$ to 0^-, a double pole moves down the imaginary axis, starting from $kb = -i$. As $\lambda \to 0^+$, the poles go to infinity in the lower complex plane approaching asymptotically the lines $xb = \pm\pi/2$. The poles are at $kb = \pm 1 - i$ for $\lambda = 1$, and reach the origin for $\lambda = 3$. As λ increases further, one pole climbs up the positive imaginary axis, thus forming a bound state whose binding energy increases with λ. The other pole moves along the negative imaginary axis, approaching the point $kb = -i$ as $\lambda \to +\infty$.

poles for odd l, and $l + 2$ such poles for even l, and only two of these can reach the origin.

E. The Limit $\lambda \to \infty$

Following the same path taken in the study of the limit $\lambda \to -\infty$, let us first look for solutions of Eq. (4.11) with $\lambda \to \infty$ and such that $|kb| \to \infty$. We want to study the behavior of $k'b$ in these solutions; Eqs. (4.21), (4.22), and (4.23) are still valid in the present limit, with the difference that now the relations between the k' and k variables are given by Eq. (4.30). We then have an infinite number of values of $x'b$ satisfying Eq. (4.23b), and these correspond by Eq. (4.30) to values of yb which increase without limit. Thus, in the kb plane the poles move along the positive and negative parts of the imaginary axis, moving away from the origin as λ increases.

As λ varies between 1 and ∞, the poles move on the real axis of the $k'b$ plane, from the points which are the roots of Eq. (4.29) to those which are the roots of Eq. (4.23). This is shown in Figs. 6(a) and 6(b) for the cases $l = 0$ and $l = 1$, respectively. The corresponding displacement of the poles along the imaginary axis of the kb plane is indicated in Figs. 5(a) and 5(b). The curves show that the binding energies of the bound-state poles increase monotonously as λ increases.

We now examine the existence of poles for which kb is kept finite as $\lambda \to \infty$. Then $k'b$ goes to zero, and Eq. (4.25) again determines the existence of such poles. Its solutions were already mentioned in Sec. 4B; there are l solutions, all in the lower half of the kb plane. We thus have that in both limits $\lambda \to -\infty$ and $\lambda \to \infty$ there are, for $l \geq 1$, l poles which remain in the finite kb plane; when $\lambda \to 0$ these poles follow the general behavior of going down the complex plane approaching asymptote lines $xb = n\pi/2$. Since these l poles do not enter the origin for $\lambda = 1$ to start moving along the imaginary axis, they belong to the group of the so-called "special" poles (which are those poles which do not reach the origin for $\lambda = 1$). We have seen that the special poles are in number of $l + 1$ for l odd and of $l + 2$ for l even. Of these, l poles will go to points with finite values of kb as $\lambda \to \pm\infty$. It is interesting to remark that the positions of these l poles are the same as those occupied by the poles of the S matrix for a hard-core potential.

We thus have that in the case of *l even*, two of the special poles enter the origin [for a value of λ given by Eq. (4.34)], one of them goes up the positive imaginary axis, while the other must move along the negative imaginary axis to *infinity*. This statement can be made since we know that of the $l + 2$ special poles of the even-l case, two must enter the origin for $\lambda = 2 + 1/l$, and as λ increases to ∞, l poles will remain in the finite kb plane; thus, the two which enter the origin must tend to infinity, one along the positive imaginary axis (thus constituting a special "late" bound state) and the other along the negative imaginary axis.

On the other hand, in the case of *odd l*, we have $l + 1$ special poles, two of which enter the origin for $\lambda = 2 + 1/l$ and remain on the imaginary axis as λ increases further. Since l poles will remain in the finite lower part of the kb plane, the special pole which enters the origin and starts moving along the negative imaginary axis will tend to a *finite* value of kb in this axis (in fact, one of the solutions of Eq. (4.26) is necessarily on the negative imaginary axis when l is odd).

We have thus traced the behavior of the special poles for all values of λ, in all cases with $l \geq 1$. Figure 8 shows the paths described by the special poles in the case $l = 1$.

In the case $l = 0$ we have, in the limits $\lambda \to \pm\infty$, two poles which remain in the finite kb plane [in the points $kb = \pm(3)^{\frac{1}{2}}/2 - (\frac{3}{2})i$]. When λ approaches 1, these two poles will be forming a double pole on the negative imaginary axis, according to Eq. (4.31). For $\lambda = 1$ this double pole will then be at $kb = -i$; as $\lambda \to 0^+$ it descends along the negative imaginary axis, and for increasing negative λ the poles will again be separated, coming up the plane guided by the asymptote lines $xb = \pm\pi/2$. For $\lambda \to -\infty$ they must be back to the location they had for $\lambda \to +\infty$. The paths described by the special poles for $l = 0$ are thus completely understood. These paths are shown in Fig. 7.

APPENDIX

We want to show how Eq. (4.24) has been derived. It represents the behavior of $yb = \mathrm{Im}\,(kb)$ at the poles when $\lambda \to -\infty$. We have that the values of $x'b$ and $y'b$ for these poles are given by Eq. (4.23a, b). Since $y'b$ tends to zero at the pole locations in this limit, we are interested in finding the behavior of $yb = y'b[1 - \lambda]^{\frac{1}{2}}$. Let us call $k_n'b = x_n'b + iy_n'b$ the location of a particular pole in the limit here considered. In the neighborhood of $k_n'b$ we have, by using a Taylor expansion up to first order,

$$k'bj_{l+1}(k'b)/j_l(k'b) \approx l + (k'b - k_n'b)R_n, \quad \text{(A1)}$$

where

$$R_n = [d(\rho j_{l+1}(\rho)/j_l(\rho))/d\rho]_{\rho = k_n'b}. \quad \text{(A2)}$$

R_n is real since $k_n'b$ is real. Substituting Eq. (A1) into the equation determining the poles, Eq. (4.11), and using asymptotic expressions for the Hankel functions of large arguments (since $|xb| \to \infty$ in the neighborhood of the poles), we obtain

$$\begin{aligned}(1 - \lambda)(l + (k'b - k_n'b)R_n)& \\ - kb(1 + (l+1)(l+2)/2kb)/i(1 + il(l+1)/2kb)& \\ + \lambda l = 0.& \quad \text{(A3)}\end{aligned}$$

Taking the dominating terms, this gives

$$l + (1 - \lambda)(k'b - k_n'b)R_n + ikb + l(l+1)/2 = 0. \quad \text{(A4)}$$

Separating real and imaginary parts, we get

$$l(l+3)/2 + (1 - \lambda)(x'b - x_n'b)R_n - yb = 0, \quad \text{(A5a)}$$
$$(1 - \lambda)y'bR_n + xb = 0. \quad \text{(A5b)}$$

This last equation gives

$$yb = -x'b/R_n \approx -x_n'b/R_n. \quad \text{(A6)}$$

Using properties of Bessel functions, we obtain that for $k_n'b$ satisfying Eq. (4.23a, b) we have

$$\begin{aligned}R_n &= k_n'b(1 - l(l+1)/(k_n'b)^2) \\ &\approx x_n'b(1 - l(l+1)/(x_n'b)^2). \quad \text{(A7)}\end{aligned}$$

Taking this into (A6), we finally obtain

$$yb = -[1 - l(l+1)/(x_n'b)^2]^{-1}, \quad \text{(A8)}$$

which is the required Eq. (4.24).

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Irreducible Representations of the Five-Dimensional Rotation Group. I*

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Explicit matrix elements are found for the generators of the group $R(5)$ in an arbitrary irreducible representation using the "natural basis" in which the representation of $R(5)$ is fully reduced with respect to the subgroup $R(4) = SU(2) \otimes SU(2)$. The technique used is based on the well-known Racah algebra. The dimension formula is derived and the invariants are found. A family of identities is established which relates various polynomials of degree four in the generators and which holds in any representation of the group.

INTRODUCTION

Recently, interest has been revived in describing the collective states of certain even–even nuclei by means of a five-dimensional isotropic harmonic oscillator arising out of the quadrupole vibrations of the nuclear surface about a spherical equilibrium shape. This model predicts[1] that the second excited state should be a degenerate triplet of angular momenta ($L^\pi = 0^+, 2^+, 4^+$) occurring at twice the excitation of the first excited state which has angular momentum and parity $L^\pi = 2^+$. As this prediction is not observed to hold in actual nuclei, the five-dimensional oscillator model is only an approximate description of the excited states of these nuclei. This description has nonetheless proved to be a convenient starting point in describing the coupling of the collective modes to the giant dipole oscillations resulting in the splitting of the giant dipole resonance.[2,3]

For the five-dimensional oscillator, only the totally symmetric irreducible representations of $SU(5)$ occur, and these may be considered to be fully reduced with respect to the subgroup $R(5)$. For application to the above problem, it is then convenient to reduce the $R(5)$ irreducible representations with respect to the physical $R(3)$.

Of course, all the main properties of the classical groups are already well known and may be found by mining in such classic works as the books by Murnaghan, Weyl, and Littlewood.[4] For practical applications, however, it is necessary to realize the irreducible representation of the group in an explicit way. This introduces the problem of labeling the states within an irreducible representation in a manner whose physical meaning is transparent. For application to the physical problem in mind, we have already indicated that the $R(5)$ representations should be explicitly reduced with respect to the physical $R(3)$ subgroup; however, it is very hard to obtain suitable explicit representation matrices directly using such a fully reduced basis. Instead, we adopt the "natural" labeling in which an irreducible representation of $R(5)$ is considered to be fully reduced with respect to its subgroup $R(4) = SU(2) \otimes SU(2)$, and a state is labeled by the particular weight of the particular irreducible representation of $R(4)$ to which it belongs. The problem of relating the natural labeling to that in which $R(5)$ is reduced with respect to the physical $R(3)$ subgroup will be the subject of our second paper.

The main original results in the present work are the development of the explicit representation matrices in the natural basis,[5] and the discussion of the well-known dimension formula and Casimir-type operators by means of our algebraic approach. So far as we know, the fourth-order identities discussed in Sec. 5 are completely new. They are analogous to those found by Pursey[6] for $SU(3)$. Much of this work was developed in embryonic form some years ago by two of us (N. K. and D. L. P.), but was not published at that time. The present treatment closely follows Pursey's treatment of $SU(3)$ in its whole-hearted exploitation of Racah algebra.

Because $R(5)$ is compact, we know that all the

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[4] F. D. Murnaghan, *The Theory of Group Representations* (Johns Hopkins Press, Baltimore, 1938); H. Weyl, *The Classical Groups* (Princeton Univ. Press, Princeton, N.J., 1946); D. E. Littlewood, *The Theory of Group Characters* (Oxford Univ. Press, London, 1940).

[5] Explicit representation matrices have been found for the generators of the rotation groups in arbitrarily many dimensions by I. M. Gel'fand and M. L. Zetlin, Dokl. Akad. Nauk. SSSR **71**, 1017 (1950). We believe our treatment is a simpler approach to the problem in the particular case of $R(5)$.

[6] D. L. Pursey, Proc. Roy. Soc. (London) **A275**, 284 (1963).

irreducible representations may be taken to be unitary and are finite-dimensional. It will not be necessary to make use of this latter property, however, since the finite dimensionality of the unitary representations will emerge from the algebraic formalism. It is perhaps of interest to note that this approach will also provide, via the unitary trick, the finite nonunitary irreducible representations of the de Sitter group.

1. CHOICE OF GENERATORS

It is well known that the generators M_{jk} of $R(5)$ satisfy the commutation relations

$$[M_{jk}, M_{lm}] = i(\delta_{jl}M_{km} + \delta_{km}M_{jl} - \delta_{jm}M_{kl} - \delta_{kl}M_{jm}), \quad (1)$$

where the indices run from 1 to 5. M_{jk} is the generalization to five dimensions of the angular-momentum tensor $\epsilon_{jkl}J_l$ in three dimensions. In particular, if A_l is a vector, then

$$[M_{jk}, A_l] = i(\delta_{jl}A_k - \delta_{kl}A_j). \quad (2)$$

It will be convenient to replace the ten linearly independent generators of Eq. (1) by linear combinations which explicitly display the $SU(2) \otimes SU(2) = R(4)$ subgroup of $R(5)$. This may be done by defining

$$\mathscr{p}_\sigma = \tfrac{1}{4}\epsilon_{\sigma jk}M_{jk} + \tfrac{1}{2}M_{\sigma 4}, \quad (3a)$$

$$\mathscr{q}_\sigma = \tfrac{1}{4}\epsilon_{\sigma jk}M_{jk} - \tfrac{1}{2}M_{\sigma 4}, \quad (3b)$$

where $\sigma, j, k = 1, 2, 3$ only, and we use the summation convention for repeated indices. Then we have the commutators

$$[\mathscr{p}_\alpha, \mathscr{p}_\beta] = i\epsilon_{\alpha\beta\gamma}\mathscr{p}_\gamma, \quad (4a)$$

$$[\mathscr{q}_\alpha, \mathscr{q}_\beta] = i\epsilon_{\alpha\beta\gamma}\mathscr{q}_\gamma, \quad (4b)$$

and

$$[\mathscr{p}_\alpha, \mathscr{q}_\beta] = 0. \quad (4c)$$

The remaining four generators are then conveniently grouped to display their transformation properties under the $SU(2) \otimes SU(2)$ subgroup generated by $\mathbf{p}$ and $\mathbf{q}$. They form a bispinor $T^{[pq]}_{\alpha\beta}$ with components

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2}\frac{1}{2}} = -2^{-\frac{1}{2}}(M_{15} + iM_{25}), \quad (5a)$$

$$T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2}-\frac{1}{2}} = 2^{-\frac{1}{2}}(M_{15} - iM_{25}), \quad (5b)$$

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2}-\frac{1}{2}} = 2^{-\frac{1}{2}}(M_{35} - iM_{45}), \quad (5c)$$

$$T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2}\frac{1}{2}} = 2^{-\frac{1}{2}}(M_{35} + iM_{45}). \quad (5d)$$

It is also convenient to replace the Cartesian generators of the two $SU(2)$ subgroups by tensors irreducible with respect to the product group. Thus we use

$$\begin{aligned} p_1 &\equiv T^{[10]}_{10} = -2^{-\frac{1}{2}}(\mathscr{p}_1 + i\mathscr{p}_2), \\ p_0 &\equiv T^{[10]}_{00} = \mathscr{p}_3, \\ p_{-1} &\equiv T^{[10]}_{-10} = 2^{-\frac{1}{2}}(\mathscr{p}_1 - i\mathscr{p}_2), \end{aligned} \quad (6)$$

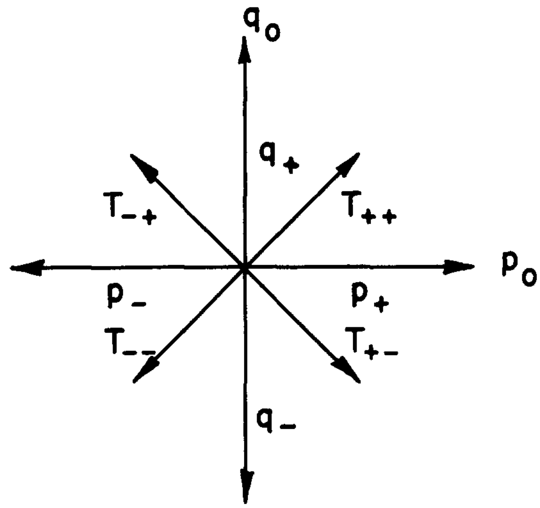

FIG. 1. The root diagram corresponding to the choice of generators of $R(5)$ given in the text. For simplicity the superscripts on the bispinor have been omitted and the $\pm\frac{1}{2}$ components denoted by $\pm$ only. Similarly, the ± 1 components of $\mathbf{p}$ and $\mathbf{q}$ are denoted $\pm$.

and similarly,

$$\begin{aligned} q_1 &\equiv T^{[01]}_{01} = -2^{-\frac{1}{2}}(\mathscr{q}_1 + i\mathscr{q}_2), \\ q_0 &\equiv T^{[01]}_{00} = \mathscr{q}_3, \\ q_{-1} &\equiv T^{[01]}_{0-1} = 2^{-\frac{1}{2}}(\mathscr{q}_1 - i\mathscr{q}_2). \end{aligned} \quad (7)$$

This choice of generators is conveniently displayed on the root diagram of Fig. 1. The commutation properties of the p's, q's, and the bispinor are then given by[7]

$$\begin{aligned} [p_\mu, p_\nu] &= 2^{-\frac{1}{2}}C(111; \nu\mu)p_{\mu+\nu}, \\ [q_\mu, q_\nu] &= 2^{-\frac{1}{2}}C(111; \nu\mu)q_{\mu+\nu}, \\ [p_\mu, q_\nu] &= 0, \\ [p_\mu, T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}] &= \frac{3^{\frac{1}{2}}}{2} C(\tfrac{1}{2}1\tfrac{1}{2}; \alpha\mu)T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha+\mu,\beta}, \end{aligned} \quad (8)$$

and

$$[q_\nu, T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}] = \frac{3^{\frac{1}{2}}}{2} C(\tfrac{1}{2}1\tfrac{1}{2}; \beta\nu)T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha,\beta+\nu}.$$

We shall not explicitly require the commutators of the elements of the bispinor among themselves. Rather we take linear combinations of these commutators with vector coupling coefficients to construct, in spherical-tensor form, the vector scalar $[T^{[\frac{1}{2}\frac{1}{2}]}, T^{[\frac{1}{2}\frac{1}{2}]}]^{[10]}_{\mu 0}$. Clearly one has

$$[T^{[\frac{1}{2}\frac{1}{2}]}, T^{[\frac{1}{2}\frac{1}{2}]}]^{[10]}_{\mu 0} = \lambda p_\mu, \quad (9)$$

and in order to find λ, we need merely consider one component, say $\mu = 1$; this yields $\lambda = -2$. Similarly,

[7] The vector coupling coefficients here are in the notation of M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

one has

$$[T^{[\frac12\frac12]}, T^{[\frac12\frac12]}]^{[01]}_{0\nu} = -2q_\nu. \quad (10)$$

We then use

$$[T^{[\frac12\frac12]}, T^{[\frac12\frac12]}]^{[10]}_{\mu 0} = 2(T^{[\frac12\frac12]}T^{[\frac12\frac12]})^{[10]}_{\mu 0}$$

to find

$$(T^{[\frac12\frac12]}T^{[\frac12\frac12]})^{[10]}_{\mu 0} = -T^{[10]}_{\mu 0} \quad (11a)$$

and

$$(T^{[\frac12\frac12]}T^{[\frac12\frac12]})^{[01]}_{0\nu} = -T^{[01]}_{0\nu}. \quad (11b)$$

2. BASIS STATES AND REDUCED MATRIX ELEMENTS

Each irreducible representation of $R(5)$ is considered to be fully reduced with respect to the product subgroup $SU(2) \otimes SU(2)$. Therefore the state vectors will bear the labels $|p\lambda q\mu\rangle$, where $p(p+1)$, λ, $q(q+1)$, and μ are the eigenvalues of p^2, p_0, q^2, and q_0, respectively. The basic problem then is to determine the ranges of p and q within a given irreducible representation of $R(5)$. We do this by finding the reduced matrix elements of the bispinor for our choice of basis states. In the notation of Fano and Racah,[8] the matrix element of $T^{[j_1j_2]}_{\alpha_1\alpha_2}$ is given by the Wigner–Eckart theorem

$$\begin{aligned}\langle p'\lambda'q'\mu'|\, T^{[j_1j_2]}_{\alpha_1\alpha_2}\, |p\lambda q\mu\rangle &= [(2p'+1)(2q'+1)]^{-\frac12} C(pj_1p';\lambda\alpha_1\lambda') \\ &\times C(qj_2q';\mu\alpha_2\mu')\, \langle p'q' \| T^{[j_1j_2]} \| pq\rangle. \quad (12)\end{aligned}$$

We have then from Eqs. (11) and (12), together with Fano and Racah's equation (15.15) and the reduced matrix elements of $T^{[10]}$ and $T^{[01]}$,

$$\begin{aligned}\sum_{p''q''} \sqrt{3}\, W(pp'\tfrac12\tfrac12;1p'')W(qq'\tfrac12\tfrac12;0q'') &\\ \times \langle p'q' \| p''q''\rangle\langle p''q'' \| pq\rangle &\\ = -\delta_{p'p}\delta_{q'q}[p(p+1)(2p+1)(2q+1)]^{\frac12} \quad (13a)\end{aligned}$$

and

$$\begin{aligned}\sum_{p''q''} \sqrt{3}\, W(pp'\tfrac12\tfrac12;0p'')W(qq'\tfrac12\tfrac12;1q'') &\\ \times \langle p'q' \| p''q''\rangle\langle p''q'' \| pq\rangle &\\ = -\delta_{p'p}\delta_{q'q}[q(q+1)(2q+1)(2p+1)]^{\frac12}, \quad (13b)\end{aligned}$$

where, in Eqs. (13), we have used Racah's notation[9] for the recoupling coefficients and have abbreviated

$$\langle p'q' \| T^{[\frac12\frac12]} \| pq\rangle \quad \text{by} \quad \langle p'q' \| pq\rangle.$$

In Eq. (13a), the left-hand side vanishes identically unless $p'' = p \pm \frac12$, $p'' = p' \pm \frac12$, $p' = p$, $p \pm 1$, and $q'' = q \pm \frac12$. First of all, take $p' = p \pm 1$, which requires that $p'' = p \pm \frac12 = p' \mp \frac12$. Then Eq. (13a) yields

$$\begin{aligned}\langle p\pm 1, q \| p \pm \tfrac12, q+\tfrac12\rangle\langle p\pm\tfrac12, q+\tfrac12 \| pq\rangle &\\ = \langle p\pm 1, q \| p\pm\tfrac12, q-\tfrac12\rangle\langle p\pm\tfrac12, q-\tfrac12 \| pq\rangle. &\quad (14)\end{aligned}$$

Let us now define

$$s = p+q, \quad t = p-q, \quad (15)$$

so that Eq. (14) becomes

$$\begin{aligned}\langle s\pm1, t\pm1 \| s\pm1, t\rangle\langle s\pm1, t \| s, t\rangle &\\ = \langle s\pm1, t\pm1 \| s, t\pm1\rangle\langle s, t\pm1 \| s, t\rangle. &\quad (16)\end{aligned}$$

Similarly, in Eq. (13b) we take $q' = q \pm 1$ and thus $q'' = q \pm \frac12 = q' \pm \frac12$ to yield

$$\begin{aligned}\langle s\pm1, t\mp1 \| s\pm1, t\rangle\langle s\pm1, t \| s, t\rangle &\\ = \langle s\pm1, t\mp1 \| s, t\mp1\rangle\langle s, t\mp1 \| st\rangle. &\quad (17)\end{aligned}$$

In Eq. (17) with the upper sign, we replace t by $t+1$ and multiply the resulting equation into Eq. (16) also with the upper sign. Then one has, after some cancellations,

$$\begin{aligned}\langle s+1, t+1 \| s+1, t\rangle\langle s+1, t \| s+1, t+1\rangle &\\ = \langle s, t+1 \| s, t\rangle\langle s, t \| s, t+1\rangle. &\quad (18)\end{aligned}$$

Since we seek unitary representations, we have so defined our generators that

$$T^{[\frac12\frac12]\dagger}_{\alpha\beta} = (-1)^{\alpha+\beta}T^{[\frac12\frac12]}_{-\alpha-\beta}, \quad (19)$$

and thus it follows from Eq. (12) that

$$\langle p'q' \| pq\rangle^* = (-1)^{p+q-p'-q'}\langle pq \| p'q'\rangle, \quad (20a)$$

or, in terms of s and t,

$$\langle s't' \| st\rangle^* = (-1)^{s-s'}\langle st \| s't'\rangle. \quad (20b)$$

Upon using Eqs. (20b) and (18), we find

$$|\langle s+1, t+1 \| s+1, t\rangle|^2 = |\langle s, t+1 \| s, t\rangle|^2 \equiv g(t), \quad (21)$$

which is clearly independent of s.

By a similar procedure, using opposite signs in Eqs. (16) and (17), we obtain

$$\begin{aligned}|\langle s+1, t+1 \| s, t+1\rangle|^2 &\\ = |\langle s+1, t \| s, t\rangle|^2 \equiv f(s). &\quad (22)\end{aligned}$$

Finally, we use Eqs. (13) with $p' = p$, $q' = q$ to obtain the induction equations for $f(s)$ and $g(t)$. From Eq. (13a), we have

$$\begin{aligned}(s+t)[f(s)+g(t)] - (s+t+2) &\\ \times [f(s-1)+g(t-1)] &\\ = -(s+t)(s+t+1)(s+t+2) &\\ \times (s-t+1), &\quad (23)\end{aligned}$$

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[8] U. Fano and G. Racah, *Irreducible Tensorial Sets* (Academic Press Inc., New York, 1959).

[9] See, for example, M. E. Rose, *Elementary Theory of Angular Momenta* (John Wiley & Sons, Inc., New York, 1957).

while Eq. (13b) yields

$$(s-t)[f(s)+g(t-1)]-(s-t+2)\times[f(s-1)+g(t)] = -(s-t)(s-t+1)(s-t+2)\times(s+t+1). \quad (24)$$

3. SOLUTION OF THE INDUCTION EQUATIONS

We may note in passing that from Eqs. (23) and (24) it is obvious that $g(t) = g(-t-1)$, but it will be unnecessary to use this symmetry property to solve these induction equations. Similarly, while it is very straightforward to use the compactness of $R(5)$ and therefore the knowledge that all unitary representations are finite, it is necessary only to use the fact that all irreducible representations may be taken as unitary. The finite dimensionality of the representations arises in a very natural way from the solution to the induction equations which themselves have come from the group algebra.

Equation (23) is rewritten as

$$\frac{f(s)+g(t)}{(s+t+2)(s-t+1)} - \frac{f(s-1)+g(t-1)}{(s+t)(s-t+1)} = -(s+t+1) = -\tfrac{1}{4}[(s+t+2)^2-(s+t)^2].$$

Therefore, noting that on both sides the second term is obtained from the first by replacing s by $s-1$ and t by $t-1$, we must have

$$f(s)+g(t) = -\tfrac{1}{4}(s+t+2)(s-t+1)\times[(s+t+2)^2+\alpha(s-t)], \quad (25)$$

where α is a function of $s-t$ as yet to be determined. Equation (24) is then rewritten as

$$\frac{f(s)+g(t-1)}{(s-t+2)(s+t+1)} - \frac{f(s-1)+g(t)}{(s-t)(s+t+1)} = -(s-t+1) = -\tfrac{1}{4}[(s-t+2)^2-(s-t)^2],$$

from which we find in a similar way that

$$(s)+g(t-1) = -\tfrac{1}{4}(s-t+2)(s+t+1)\times[(s-t+2)^2+\beta(s+t)]. \quad (26)$$

In Eq. (26) we replace t by $t+1$ and compare the result with Eq. (25). This gives

$$\beta(s+t) = (s+t+1)^2+2\gamma,$$
$$\alpha(s-t) = (s-t+1)^2+2\gamma. \quad (27)$$

On substituting these results into Eq. (25) or Eq. (26), we find

$$f(s)+g(t) = -\tfrac{1}{4}(s+t+2)(s-t+1)\times[(s+t+2)^2+(s-t+1)^2+2\gamma] = -\tfrac{1}{2}[(s+\tfrac{3}{2})^2-(t+\tfrac{1}{2})^2]\times[(s+\tfrac{3}{2})^2+(t+\tfrac{1}{2})^2+\gamma].$$

Therefore,

$$f(s) = -\tfrac{1}{2}\{(s+\tfrac{3}{2})^2[(s+\tfrac{3}{2})^2+\gamma]+\delta\} \quad (28a)$$

and

$$g(t) = +\tfrac{1}{2}\{(t+\tfrac{1}{2})^2[(t+\tfrac{1}{2})^2+\gamma]+\delta\}. \quad (28b)$$

Let us now introduce $x = (s+\frac{3}{2})^2$ and $y = (t+\frac{1}{2})^2$. Since $x > 0$, $y \geq 0$, and $|t| \leq |s|$, we have

$$0 \leq y < x. \quad (29)$$

Furthermore, since $f(s)$ and $g(t)$ are intrinsically positive, we have

$$-\tfrac{1}{2}x^2-\tfrac{1}{2}\gamma x-\tfrac{1}{2}\delta \geq 0 \quad (30a)$$

and

$$\tfrac{1}{2}y^2+\tfrac{1}{2}\gamma y+\tfrac{1}{2}\delta \geq 0. \quad (30b)$$

From Eq. (30a) it follows that x must lie between the roots of $x^2+\gamma x+\delta = 0$ and, from Eq. (30b), y must lie outside the roots. Thus we have

$$0 \leq y \leq -\tfrac{1}{2}\gamma-(\tfrac{1}{4}\gamma^2-\delta)^{\frac{1}{2}} \leq x \leq -\tfrac{1}{2}\gamma+(\tfrac{1}{4}\gamma^2-\delta)^{\frac{1}{2}}. \quad (31)$$

It therefore follows that $\gamma \leq 0$ and $0 \leq \delta \leq (\gamma/2)^2$.

It is clear that the bispinor has the ladder property for s and t, in that it steps either s or t up or down by 1. Hence y must reach its upper bound, for only then will $g(t)$ vanish and terminate the ladder. Similarly, x must attain its upper bound. Thus

$$f(s_{\max}) = 0 \quad (32a)$$

and

$$g(t_{\max}) = 0. \quad (32b)$$

Let us denote the upper bounds of s and t by l and k, respectively. Then we have

$$(l+\tfrac{3}{2})^2 = -\tfrac{1}{2}\gamma+(\tfrac{1}{4}\gamma^2-\delta)^{\frac{1}{2}} \quad (33a)$$

and

$$(k+\tfrac{1}{2})^2 = -\tfrac{1}{2}\gamma-(\tfrac{1}{4}\gamma^2-\delta)^{\frac{1}{2}}, \quad (33b)$$

from which it follows that

$$\gamma = -(l+\tfrac{3}{2})^2-(k+\tfrac{1}{2})^2 \quad (34)$$

and

$$\delta = (l+\tfrac{3}{2})^2(k+\tfrac{1}{2})^2. \quad (35)$$

Using Eqs. (34) and (35) with Eqs. (28), we find

$$f(s) = \tfrac{1}{2}(l-s)(l+s+3)(s-k+1)(s+k+2) \quad (36)$$

and

$$g(t) = \tfrac{1}{2}(l-t+1)(l+t+2)(k-t)(k+t+1). \quad (37)$$

To recapitulate we have

$$\begin{aligned} f(s) &= |\langle s+1, t \| T^{[\frac{1}{2}\frac{1}{2}]} \| s, t\rangle|^2 \\ &= |\langle p+\tfrac{1}{2}, q+\tfrac{1}{2} \| T^{[\frac{1}{2}\frac{1}{2}]} \| pq\rangle|^2, \\ g(t) &= |\langle s, t+1 \| T^{[\frac{1}{2}\frac{1}{2}]} \| s, t\rangle|^2 \\ &= |\langle p+\tfrac{1}{2}, q-\tfrac{1}{2} \| T^{[\frac{1}{2}\frac{1}{2}]} \| pq\rangle|^2, \\ s &= p+q, \quad t = p-q, \end{aligned} \tag{38}$$

and the solution to the induction equations is completed once we adopt the phase convention that the reduced matrix elements are the positive square roots of $f(s)$ and $g(t)$.

4. CHARACTERIZATION OF THE IRREDUCIBLE REPRESENTATIONS AND THE DIMENSION FORMULA

We have solved the induction equations in terms of the maximum values l and k of s and t in the representation. We must still determine the permissible values of l and k. We have made use of the essential nonnegative character of $f(s)$ and $g(t)$, which implies that the induction equation shall not lead to values of x greater than $(l + \frac{3}{2})^2$ nor less than $(k + \frac{1}{2})^2$ nor to values of y greater than $(k + \frac{1}{2})^2$. From the definitions of l and k, it is clear that $l \geq k \geq 0$. From the necessary symmetry of the irreducible representations under interchange of p and q, it is clear that $-k \leq t \leq k$ so that k must be an integer or half-integer; and since $l - k$ must be an integer, l is correspondingly an integer or half integer.

Thus we conclude that *an irreducible representation of $R(5)$ is characterized by two nonnegative numbers (l, k) such that both are either integers or half-integers and $l \geq k$. For given (l, k), s ranges from k to l by steps of* 1, *and t from $-k$ to k by steps of* 1.

We may express these results in terms of p and q as follows: *In an irreducible representation (l, k) of $R(5)$, p and q range from* 0 *by steps of $\frac{1}{2}$ to $\frac{1}{2}(k + l)$. For a fixed value of q, p ranges from $|k - q|$ by steps of* 1 *to the minimum of $[l - q, k + q]$. For a fixed value of p, q ranges from $|k - p|$ by steps of* 1 *to the minimum of* $[l - p, k + p]$.

The dimension of an irreducible representation (l, k) may be easily computed by summing $(2p + 1)(2q + 1)$ over the possible simultaneous values of p and q. Alternatively, we may sum $(s + t + 1)(s - t + 1)$ over the permissible values of s and t to obtain

$$d(l, k) = \tfrac{1}{6}(2k + 1)(2l + 3)(l + k + 2)(l - k + 1). \tag{39}$$

The irreducible representations of $R(5)$ may also be characterized using the well-known isomorphism with $Sp(4)$[10] for which the irreducible representations are put into one to one correspondence with the two-rowed Young tableaux labeled by σ_1 and σ_2, which are the numbers of boxes in the first and second rows, respectively. This corresponds to classifying tensors under $Sp(4)$ according to the symmetry properties of their indices. The vector representation of $Sp(4)$ is the spinor representation of $R(5)$ and the connection between the characterizations is

$$\sigma_1 = l + k, \quad \sigma_2 = l - k. \tag{40}$$

The substitution of Eqs. (40) into Eq. (39) gives Weyl's result.[11]

In yet another characterization, based on weight diagrams, Speiser[12] labels the irreducible representations by (L_1, L_2), which are related to σ_1, σ_2, l, and k by

$$L_1 = \sigma_2 = l - k, \quad L_2 = \sigma_1 - \sigma_2 = 2k. \tag{41}$$

Finally, we make connection with the characterization given by Hecht[13] and by Parikh,[14] in which the labels are (p_m, q_m) which are the values of λ and μ for the state of maximum weight. The relationships are

$$p_m = \tfrac{1}{2}(l + k), \quad q_m = \tfrac{1}{2}(l - k) \tag{42}$$

or

$$p_m + q_m = s_{\max} = l, \quad p_m - q_m = t_{\max} = k. \tag{43}$$

5. INVARIANTS

The most direct approach to finding the invariants is undoubtedly to directly construct operators which commute with all the group generators. The matrix elements of these operators must then be expressible as a function of only l and k. Since there are but two numbers required to characterize a representation, there exist only two independent invariants. For the group $R(5)$ there will be a second-order and a fourth-order invariant since the third-order invariant $M_{ij}M_{jk}M_{ki}$ obviously vanishes. Thus, we could construct $M_{ij}M_{ji}$ and $M_{ij}M_{jk}M_{kl}M_{li}$ directly. However, this is not the most convenient way to proceed.

We shall find it most convenient to define the operator

$$T^2 = -(T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]})^{[00]}. \tag{44}$$

Then, since this obviously commutes with p^2 and q^2, we need consider only its diagonal reduced matrix

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[10] G. Racah, "Lectures on Lie Groups," *Group Theoretic Concepts and Methods in Elementary Particle Physics*, F. Gürsey, Ed. (Gordon and Breach, Science Publishers, Inc., New York, 1964).

[11] H. Weyl, *The Classical Groups* (Princeton Univ. Press, Princeton, N.J., 1946).

[12] D. Speiser, "Theory of Compact Lie Groups and Some Applications to Elementary Particle Physics," *Group Theoretic Concepts and Methods in Elementary Particle Physics*, F. Gürsey, Ed. (Gordon and Breach Science Publishers, Inc., New York, 1964).

[13] K. T. Hecht, Nucl. Phys. **63**, 177 (1965).

[14] J. C. Parikh, Nucl. Phys. **63**, 214 (1965).

elements, which are given by

$$\langle (lk)pq \| T^2 \| (lk)pq\rangle = -\sum_{p'q'} W(pp\tfrac{1}{2}\tfrac{1}{2}; 0p')W(qq\tfrac{1}{2}\tfrac{1}{2}; 0q') \times \langle pq \| p'q'\rangle\langle p'q' \| pq\rangle. \quad (45)$$

From this, together with Eqs. (20a), (21), and (22), we obtain

$$\langle (lk)pq \| T^2 \| (lk)pq\rangle = \tfrac{1}{2}[(2p+1)(2q+1)]^{-\frac{1}{2}} \times \{f(s) + f(s-1) + g(t) + g(t-1)\}. \quad (46)$$

Then, using Eqs. (36) and (37), we find after some simplification

$$\langle (lk)pq \| T^2 \| (lk)pq\rangle = [(2p+1)(2q+1)]^{\frac{1}{2}} \times \{a(l,k) - p(p+1) - q(q+1)\}, \quad (47)$$

in which $a(l, k)$ is defined by

$$2a(l,k) = l(l+3) + k(k+1). \quad (48)$$

Therefore, we obtain an invariant A^2 defined by

$$A^2 = T^2 + p^2 + q^2, \quad (49)$$

whose value in the representation (l, k) is

$$A^2 = a(l,k). \quad (50)$$

This is essentially the second-order Casimir operator.

To obtain the fourth-order invariant we construct two bilinear operators

$$\tau^{[11]} \equiv (T^{[10]}T^{[01]})^{[11]}, \quad (51a)$$

or

$$\tau^{[11]}_{\alpha\beta} = p_\alpha q_\beta, \quad (51b)$$

and

$$T^{[11]} = (T^{[\frac{1}{2}\frac{1}{2}]}T^{[\frac{1}{2}\frac{1}{2}]})^{[11]}. \quad (52)$$

Then we consider the reduced matrix elements of

$$B^4 \equiv (\tau^{[11]}T^{[11]})^{[00]}. \quad (53)$$

Since

$$\langle (lk)p'q' \| \tau^{[11]} \| (lk)pq\rangle = \delta_{pp'}\delta_{qq'}[p(p+1)(2p+1)q(q+1)(2q+1)]^{\frac{1}{2}}, \quad (54)$$

we need only consider the diagonal elements of $T^{[11]}$, which are

$$\langle (lk)pq \| T^{[11]} \| (lk)pq\rangle = -\tfrac{1}{2}[p(p+1)(2p+1)q(q+1)(2q+1)]^{-\frac{1}{2}} \times \sum_{p'q'} [p(p+1) - p'(p'+1) + \tfrac{3}{4}] \times [q(q+1) - q'(q'+1) + \tfrac{3}{4}]\, |\langle p'q' \| pq\rangle|^2. \quad (55)$$

Then we find

$$\langle (kl)pq \| B^4 \| (kl)pq\rangle = \tfrac{1}{6}[(2p+1)(2q+1)]^{\frac{1}{2}} \times \{p(q+1)g(t) + q(p+1)g(t-1) - pqf(s) - (p+1)(q+1)f(s-1)\}. \quad (56)$$

To reduce this it is convenient to note that we may write

$$g(t) = \tfrac{1}{2}\{b(l,k) - 2t(t+1)a(l,k) + (t-1)t(t+1)(t+2)\} \quad (57a)$$

and

$$f(s) = -\tfrac{1}{2}\{b(l,k) - 2(s+1)(s+2)a(l,k) + s(s+1)(s+2)(s+3)\}, \quad (57b)$$

in which

$$b(l,k) = (l+1)(l+2)k(k+1). \quad (58)$$

Using Eqs. (57), after some simplification one has

$$\langle (kl)pq \| B^4 \| (kl)pq\rangle = \tfrac{1}{12}[(2p+1)(2q+1)]^{\frac{1}{2}}\{b(l,k) - 2a(l,k)[p(p+1) + q(q+1)] + p^2(p+1)^2 + q^2(q+1)^2 - 2p(p+1) - 2q(q+1) + 6p(p+1)q(q+1)\}. \quad (59)$$

Therefore, the operator

$$M^4 = 12B^4 + (p^2 - q^2)^2 + 2(p^2 + q^2)(T^2 + 1) \quad (60)$$

is invariant and in the representation (l, k) has the value

$$M^4 = b(l,k) = (l+1)(l+2)k(k+1). \quad (61)$$

At first sight, the invariant of Eqs. (60) and (61) seems curious in that the usual way of forming the second invariant operator would have it symmetric in all the Cartesian generators and of fourth order. M^4 contains second-order terms and does not involve

$$T^4 \equiv (T^{[11]}T^{[11]})^{[00]}. \quad (62)$$

One would therefore expect that there exists an identity relating T^4 to p^2, q^2, and possibly other quantities. This indeed is the case as may be readily demonstrated by considering

$$T_{(\alpha_1\beta_1)(\alpha_2\beta_2)(\alpha_3\beta_3)(\alpha_4\beta_4)} \equiv T_{\alpha_1\beta_1}[T_{\alpha_2\beta_2}, T_{\alpha_3\beta_3}]_+ T_{\alpha_4\beta_4}, \quad (63)$$

where the $T_{\alpha_i\beta_i}$ of Eq. (63) are indicated without superscripts and may be any one of the irreducible tensor generators of the group. From this tensor, which is clearly symmetric on its second and third pairs of indices, we may form irreducible tensors. For example, we may couple the first two pairs to $[j_1, \lambda_1]$ and the second two pairs to $[j_2, \lambda_2]$ and then couple the result to $[JL]$. We may also couple the first and third pairs to $[j_1'\lambda_1']$, the second and fourth pairs to $[j_2', \lambda_2']$ and then couple to $[JL]$. Because of the explicit symmetry between the second and third pairs of indices, the two couplings produce the same sets of irreducible tensors. Recoupling leads to the identities

$$T^{[j_1\lambda_1, j_2\lambda_2; JL]} = \sum_{\substack{j_1', \lambda_1', \\ j_2', \lambda_2'}} [(2j_1+1)(2\lambda_1+1)(2j_2+1)(2\lambda_2+1) \times (2j_1'+1)(2\lambda_1'+1)(2j_2'+1)(2\lambda_2'+1)]^{\frac{1}{2}} \times \begin{Bmatrix} p_1 p_2 j_1 \\ p_3 p_4 j_2 \\ j_1' j_2' J \end{Bmatrix} \begin{Bmatrix} q_1 q_2 \lambda_1 \\ q_3 q_4 \lambda_2 \\ \lambda_1 \lambda_2' L \end{Bmatrix} T^{[j_1'\lambda_1', j_2'\lambda_2'; JL]}, \tag{64}$$

where { } is the usual $9j$ symbol. We are specifically interested in the tensor $T^{[11,11;00]}$ formed from four bispinor components. From Eq. (64) we find

$$T^{[11,11;00]} = \sum_{j\lambda} [(2j+1)(2\lambda+1)]^{\frac{1}{2}}(-1)^{j+\lambda} \times W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}; 1j) W(\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}; 1\lambda) T^{[j\lambda, j\lambda; 00]}. \tag{65}$$

When we use explicit forms for the Racah coefficients, Eq. (65) becomes

$$T^{[11,11;00]} = 9T^{[00,00;00]} - 3\sqrt{3}(T^{[10,10;00]} + T^{[01,01;00]}). \tag{66}$$

Now, from Eq. (63) with all $T_{\alpha\beta} \equiv T_{\alpha\beta}^{[\frac{1}{2}\frac{1}{2}]}$, we have

$$T^{[11,11;00]} = T^4, \tag{67a}$$

$$T^{[00,00;00]} = (T^2)^2, \tag{67b}$$

$$T^{[10,10;00]} = -p^2, \tag{67c}$$

and

$$T^{[01,01;00]} = -q^2. \tag{67d}$$

Equations (67) together with Eq. (66) provide the identity we seek, namely,

$$T^4 = 9(T^2)^2 + 3\sqrt{3}(p^2 + q^2). \tag{68}$$

Thus, the invariant of Eq. (60) can be written in terms of fourth-order quantities as

$$M^4 = 12B^4 + (p^2 - q^2)^2 + 2(p^2 + q^2)T^2 + \tfrac{2}{9}\sqrt{3}[T^4 - 9(T^2)^2]. \tag{69}$$

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Irreducible Representations of the Five-Dimensional Rotation Group. II*

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(Received 18 December 1967)

A systematic study is made of the relationship between the generators of $R(5)$ expressed in the "natural basis," as discussed in I [J. Math. Phys. **9**, 1224 (1968)], and the same generators in the "physical basis" in which representations of $R(5)$ are fully reduced with respect to the physical three-dimensional rotation group. In this paper, attention is confined to the traceless symmetric tensors of $R(5)$ which are the representations appropriate to the discussion of quadrupole vibrations of the nuclear surface. For these representations, one quantum number in addition to the angular momentum and its projection is required to specify a state within a representation. The required extra label is found through the definition of "intrinsic states" in the natural basis, and a complete set of states in the physical basis is projected out of these intrinsic states by integrations over the physical rotation group manifold. Members of this set of physical states are not orthonormal; however, the overlap integrals are presented in two simple algebraic forms convenient for computer programming. The construction of the explicit representation matrices for the generators of $R(5)$ is completed by giving the reduced matrix elements of the octopole generator between physical states in terms of the overlap integrals.

INTRODUCTION

In I[1] the irreducible representations of $R(5)$ were built up directly from the generator algebra in a manner closely analogous to that usually done for $SU(2)$. For this purpose, the irreducible representations of $R(5)$ were reduced with respect to the subgroup $R(4) = SU(2) \otimes SU(2)$. We shall call the state labeling so developed the *natural labeling*. Unfortunately, neither of these two $SU(2)$ subgroups corresponds to the physical angular momentum. For physical application it is essential that the irreducible representations of $R(5)$ be decomposed into irreducible representations of the physical $R(3)$.

The particular physical application we have in mind is the five-dimensional harmonic oscillator which has been used[2] to describe quadrupole vibrations of the

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[1] N. Kemmer, D. L. Pursey, and S. A. Williams, J. Math. Phys. **9**, 1224 (1968) (preceding paper).

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nuclear surface about a spherical equilibrium shape. Such a description is, of course, only very approximate, but it has proved to be a convenient starting point for describing the coupling of these nuclear surface oscillations to the oscillations of the giant dipole resonance.[3,4] The state functions for the five-dimensional isotropic harmonic oscillator form the bases for the totally symmetric irreducible representations of $SU(5)$ and these may be considered to be fully reduced with respect to the subgroup $R(5)$. In this paper, we shall therefore confine ourselves to the problem of the decomposition of the symmetric irreducible representations of $R(5)$ with respect to $R(3)$.

The irreducible representations of $R(3)$ contained in a given irreducible representation of $R(5)$ may of course be obtained by the well-known methods of the reduction of product representations for both $R(5)$ and $R(3)$.[5,6] This technique, while useful for some purposes, does not allow one to perform detailed calculations using the basis functions involved, since the method does not yield explicit representations matrices for all the generators. A further difficulty arises in that, within a given irreducible representation of $R(5)$, a particular irreducible representation of $R(3)$ may occur more than once. When we have specified the generators, it will become clear that no proper subgroup of $R(5)$ [apart from the physical $R(3)$ itself] contains the particular $R(3)$ (the physical angular-momentum group) in which we are interested. Therefore, we must seek additional labels, which cannot be obtained by the aforementioned technique, to completely specify the basis functions. For the symmetric irreducible representations of $R(5)$ we shall find that only one additional label is necessary.

This additional label is obtained in a manner closely analogous to that of Elliott[7] for $SU(3)$. Specifically, the $R(5)$–$R(3)$ basis functions will be projected from a small subset of the natural basis functions by Hill–Wheeler[8]-type integrals. This will lead, as it did in Elliott's case, to $R(5)$–$R(3)$ basis functions which are not orthogonal. The fact that the solution to the problem is simpler in terms of nonorthogonal functions is not too surprising in view of Elliott's work and Racah's[9] comments about the $SU(3)$ problem. This lack of orthogonality presents no serious difficulties in general, and for convenience for machine coding, the set of linearly independent functions given could of course be orthogonalized.

In Sec. 1, we shall give the connection between the natural generators and those which explicitly exhibit the $R(3)$ subgroup. This is done in preparation for Sec. 2, in which we shall give a formula for the irreducible representations of $R(3)$ which occur in a given symmetric irreducible representation of $R(5)$. This formula serves to introduce the additional quantum number in an empirical way. In Sec. 3, we shall relate this additional quantum number to the natural basis functions and show that only a small subset of the natural basis function are required for projecting out the nonorthonormal $R(5)$–$R(3)$ basis functions. In Sec. 4, we will explicitly determine the normalization and overlap integrals for the $R(5)$–$R(3)$ basis functions. These quantities are used in Sec. 5, in which we shall give expressions for the matrix elements of the group generators expressed in the $R(5)$–$R(3)$ basis.

1. GENERATORS

In I we utilized the natural generators of $R(5)$, namely, p_μ, q_ν, and $T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}$. The p_μ and q_ν are the generators of the two commuting $SU(2)$ subgroups and the $T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}$ are the remaining generators which are expressed as a bispinor under the product group $SU(2)\otimes SU(2)$. These generators satisfy the commutation rules:

$$\begin{aligned}
[p_\mu, p_\nu] &= -\sqrt{2}\, C(111;\mu\nu)p_{\mu+\nu},\\
[q_\mu, q_\nu] &= -\sqrt{2}\, C(111;\mu\nu)q_{\mu+\nu},\\
[p_\mu, q_\nu] &= 0,\\
[p_\mu, T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}] &= \tfrac{1}{2}\sqrt{3}\, C(\tfrac{1}{2}1\tfrac{1}{2};\alpha\mu)T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha+\mu,\beta},
\end{aligned} \tag{1}$$

and

$$[q_\nu, T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha\beta}] = \tfrac{1}{2}\sqrt{3}\, C(\tfrac{1}{2}1\tfrac{1}{2};\beta\nu)T^{[\frac{1}{2}\frac{1}{2}]}_{\alpha,\beta+\nu}.$$

In Eqs. (1), the $SU(2)$ Clebsch–Gordan coefficients are in the notation of Rose.[10]

The $R(5)$ generators may also be grouped as the generators of $R(3)$ together with a third-rank irreducible tensor under $R(3)$. These we write as J_μ and $Q^{[3]}_\nu$. The latter is abbreviated as Q_ν. When expressed in spherical tensor form, these generators satisfy the commutation relationships:

$$\begin{aligned}
[J_\mu, J_\nu] &= -\sqrt{2}\, C(111;\mu\nu)J_{\mu+\nu},\\
[J_\mu, Q_\nu] &= -2\sqrt{3}\, C(133;\mu\nu)Q_{\mu+\nu},
\end{aligned} \tag{2}$$

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[9] G. Racah, "Lectures on Lie Groups," *Group Theoretical Concepts and Methods in Elementary Particle Physics* (Gordon and Breach, Science Publ., Inc., New York, 1964).
[10] M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, New York, 1957).

and

$$[Q_\mu, Q_\nu] = -2\sqrt{7}\, C(331; \mu\nu) J_{\mu+\nu} + \sqrt{6}\, C(333; \mu\nu) Q_{\mu+\nu}.$$

From the commutation algebra among the generators we may identify the natural basis generators in terms of the $R(5)$–$R(3)$ basis generators. We find[11]

$$p_1 = 10^{-\frac{1}{2}} Q_3, \tag{3a}$$

$$p_0 = 10^{-1}(3J_0 - Q_0), \tag{3b}$$

and

$$p_{-1} = 10^{-\frac{1}{2}} Q_{-3}; \tag{3c}$$

also

$$q_1 = 5^{-1}(J_1 + \tfrac{1}{2}6^{\frac{1}{2}} Q_1), \tag{4a}$$

$$q_0 = 10^{-1}(J_0 + 3Q_0), \tag{4b}$$

$$q_{-1} = 5^{-1}(J_{-1} + \tfrac{1}{2}6^{\frac{1}{2}} Q_{-1}); \tag{4c}$$

and, finally,

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2}\frac{1}{2}} = 5^{-\frac{1}{2}} Q_2, \tag{5a}$$

$$T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2}\frac{1}{2}} = 5^{-1}(3^{\frac{1}{2}} J_{-1} - 2^{\frac{1}{2}} Q_{-1}), \tag{5b}$$

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2},-\frac{1}{2}} = -5^{-1}(3^{\frac{1}{2}} J_1 - 2^{\frac{1}{2}} Q_1), \tag{5c}$$

and

$$T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2},-\frac{1}{2}} = -5^{-\frac{1}{2}} Q_{-2}. \tag{5d}$$

In Eqs. (5), the subscripts on the bispinor are, as usual, p, q ordered. From the second of Eqs. (3) and (4) we have the primary equation which relates the natural basis to the $R(5)$–$R(3)$ basis; namely,

$$J_0 = 3p_0 + q_0. \tag{6}$$

2. BASIS FUNCTIONS

In I we showed that the irreducible representations of $R(5)$ were characterized by two nonnegative numbers (l, k) which are either both integers or both half integers. For given (l, k), $s \equiv p + q$ ranges from k to l by steps of 1 and $t \equiv p - q$ from $-k$ to k by steps of 1. The symmetric representations of $R(5)$ are $(l, 0)$, in which case, since $k = 0$, $p = q$. Then, since $s = 0, 1, 2, \cdots, l$, we have that for the symmetric representations $(l, 0)$

$$p = q = 0, \tfrac{1}{2}, 1, \ldots\ldots, \tfrac{1}{2}l. \tag{7}$$

In general, to completely specify a basis function for an irreducible representation of $R(5)$ requires six labels. In the natural basis these are (l, k), p, q, λ, μ. The p and q label the two commuting $SU(2)$ subgroups, and λ and μ are the eigenvalues of p_0 and q_0, respectively. The (l, k) are related to the eigenvalues of the two invariant operators constructed from the generators of $R(5)$. These operators and their eigenvalues are

$$A^2 \equiv -[T^{[\frac{1}{2}\frac{1}{2}]} T^{[\frac{1}{2}\frac{1}{2}]}]^{[00]} + p^2 + q^2 = \tfrac{1}{2}[l(l+3) + k(k+1)]$$

and

$$\begin{aligned} M^4 \equiv\ & 12[[pq]^{[11]}[T^{[\frac{1}{2}\frac{1}{2}]} T^{[\frac{1}{2}\frac{1}{2}]}]^{[11]}]^{[00]} + (p^2 - q^2)^2 \\ & + 2(p^2 + q^2)\{-[T^{[\frac{1}{2}\frac{1}{2}]} T^{[\frac{1}{2}\frac{1}{2}]}]^{[00]} + 1\} \\ =\ & (l+1)(l+2)k(k+1). \end{aligned} \tag{8}$$

For the symmetric representations, only the second-order $R(5)$ invariant is, in general, nonzero and its value from the first of Eq. (8) is $\frac{1}{2}l(l+3)$. We also require p, λ, and μ—four labels in all. Thus for the symmetric representations we may denote the natural basis functions as $\chi(lp\lambda\mu)$.

In the $R(5)$–$R(3)$ basis we shall also require four labels for the symmetric-representation basis functions. Three of these are l, J, and M, which are the simultaneous eigenvalues of

$$\begin{aligned} A^2 &= \tfrac{1}{2}l(l+3), \\ J^2 &= J(J+1), \\ J_0 &= M. \end{aligned} \tag{9}$$

From Eqs. (2), it is clear that no proper subgroup of $R(5)$ contains $R(3)$ as a proper subgroup. Further, although Q^2 commutes with A^2, J^2, and J_0, $J^2 + Q^2$ is essentially A^2, and hence Q^2 does not provide a new label. We shall label the $R(5)$–$R(3)$ basis functions as $\psi(l\nu JM)$ where the additional label ν is as yet unspecified.

We may determine the possible values of J within a representation by a method similar to the well-known derivation of the Clebsh–Gordan series for $R(3)$. The procedure is illustrated in Fig. 1 for the representations $(1, 0)$ and $(6, 0)$. A grid of points (λ, μ), where λ and μ are the eigenvalues of p_0 and q_0, respectively, is set up, and at each point of the grid we write the degeneracy of the corresponding pair (λ, μ) of eigenvalues. We next draw the lines $3\lambda + \mu =$ const $= M$ and label each line by $[M, d(M)]$, where $d(M)$ is the total degeneracy of the eigenvalue M of J_0. The possible values of J, together with their degeneracies, are then found by a simple counting procedure: the angular momentum J occurs $d(J) - d(J+1)$ times.

From Fig. 1 it is clear that the extra quantum number ν is indeed necessary to distinguish between the two occurrences of $J = 6$ in the irreducible

[11] To obtain Eqs. (3)–(5) it is easiest first to convince oneself of the truth of Eq. (6). Apart from factors, this is sufficient to establish Eqs. (3a), (3c), (5a), and (5d). The form (3b) and the numerical factors in Eqs. (3) follow from the $SU(2)$ commutation rules obeyed by p_μ. Equations (3b) and (6) then yield (4b). Equation (5b) is derived from Eqs. (5a), (3c) and similarly Eq. (5c) comes from Eqs. (5d) and (3a). The commutation properties of the bispinor fix the constants in Eqs. (5) and also yield Eqs. (4).

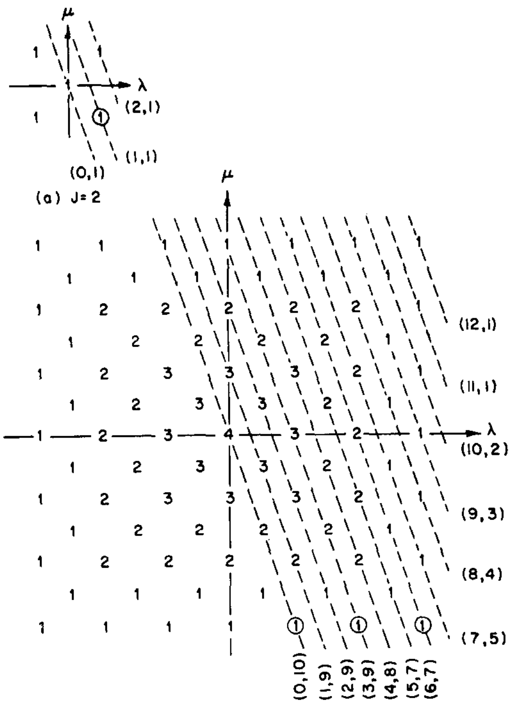

FIG. 1. Degeneracy diagrams for the (a) (1, 0) and (b) (6, 0) irreducible representations of $R(5)$. The figure is discussed in the main text.

representation (6, 0). Here we shall introduce a suitable extra label and state the general rule for the range of possible J values associated with the label ν in the irreducible representation $(l, 0)$. The proof of the rule will be postponed until the next section.

We define *intrinsic states* $\chi(l, \nu)$ by

$$\chi(l, \nu) \equiv \chi(l, \tfrac{1}{2}l, \tfrac{1}{2}l - \nu, -\tfrac{1}{2}l), \tag{10}$$

where

$$\nu = 0, 1, 2, \cdots, [l/3] \tag{11}$$

and $[l/3]$ is the integral part of $l/3$. The points corresponding to intrinsic states are circled in Fig. 1. We next introduce the numbers

$$K = l - 3\nu, \tag{12}$$

which are the values of M for the intrinsic states. K takes on the values l, $l - 3$, $l - 6$, $\cdots$, 0 or 1 or 2. Then, corresponding to any ν or, equivalently, any K, the possible values of J are

$$J = 2K, 2K - 2, 2K - 3, \cdots, K; \tag{13}$$

that is, J can take on all values from K through $2K$ except for $2K - 1$. Furthermore, use of the label ν is sufficient to resolve the degeneracy in J.

We are now faced not only with the problem of proving this general result, but also with that of constructing from the intrinsic states $\chi(l, \nu)$ a complete (but not necessarily orthonormal) set of states $\psi(l\nu JM)$. We shall discuss these problems in the next section, and content ourselves here with noting that, as proved in Appendix A, our general rule reproduces the dimension formula

$$d(l, 0) = \tfrac{1}{6}(l + 1)(l + 2)(2l + 3) \tag{14}$$

found in I.

3. MAIN THEOREM

In this section, we shall prove the general result expressed in Eqs. (11) and (13). To do so, we shall define the states

$$\psi(l\nu JM) = \int D^{J*}_{M,K}(\Omega)\chi_\Omega(l, \nu)\, d\Omega, \tag{15}$$

where

$$\nu = 0, 1, 2, \cdots, [l/3],$$
$$K = l - 3\nu,$$

and

$$J = 2K, 2K - 2, 2K - 3, \cdots, K;$$

$\chi(l, \nu) \equiv \chi(l, \frac{1}{2}l, \frac{1}{2}l - \nu, -\frac{1}{2}l)$ and $D^J_{M,K}(\Omega)$ is an ordinary rotation matrix; finally, the integral is the invariant integral over the group manifold of $R(3)$. Thus Eq. (15) defines $\psi(l\nu JM)$ as the state of angular momentum J, z component of angular momentum M, projected out of the intrinsic state $\chi(l, \nu)$ by the Hill–Wheeler[8] technique. We now state our main theorem:

Theorem: The functions $\psi(l\nu JM)$, defined by Eq. (15), span the representation space of the irreducible representation $(l, 0)$ of $R(5)$.

Before we start on the proof, we note that the results in the previous section on the possible values of J follow as a trivial corollary.

We first prove the following:

Lemma: Any angular momentum J which is not represented by at least one of the functions $\psi(l\nu JM)$ defined by Eq. (15) is entirely absent from the irreducible representation $(l, 0)$ of $R(5)$.

This lemma is of course much weaker than the result asserted in the last section and implied by the main theorem and is consequently easier to prove. To establish the lemma, we must determine which values of J are absent according to Eqs. (12) and (13), and then verify that these J values are indeed missing from the irreducible representation $(l, 0)$ of $R(5)$. Two cases arise:

(a) $l = 3n$: The missing values of J are $J = 1, 2, 5$, $2l - 1$, and $J > 2l$.

(b) $l \neq 3n$: The missing values of J are $J = 0, 1, 3, 2l-1$, and $J > 2l$.

From the degeneracy diagrams introduced in the previous section, it is clear that $J > 2l$ and $J = 2l - 1$ are indeed absent from the irreducible representation $(l, 0)$. It remains to show that $J = 1, 2, 5$ are missing if $l = 3n$ and $J = 0, 1, 3$ are missing if $l \neq 3n$. We notice that, to conclude that a particular value of J is missing, it is sufficient to show that $d(J) = d(J+1)$ in the notation of the previous section. A detailed formal proof of the lemma would be tedious and not particularly illuminating; instead we shall sketch the bare bones of a proof, making full use of insights gained from the degeneracy diagrams.

By comparing the degeneracies of the pairs of eigenvalues (p_0, q_0) and $(p_0 + 1, q_0)$ it is clear that, for sufficiently small M,

$$d(M) - d(M+3) = [M/2] + 1, \tag{16}$$

where, as before, $[M/2]$ is the integral part of $M/2$. In particular,

$$d(0) - d(3) = 1,$$
$$d(1) - d(4) = 1,$$
$$d(2) - d(5) = 2,$$
$$d(3) - d(6) = 2.$$

From this it follows that one, but only one, of the J values 0, 1, 2 occurs and that this J value is nondegenerate. The same conclusion applies to the possible values 1, 2, and 3 of J. Of the set of values 2, 3, and 4, either only one occurs with degeneracy 2 or else two distinct values occur; a similar conclusion applies to the values 3, 4, and 5.

Now, either $J = 0$ occurs in the representation $(l, 0)$ or it does not. If $J = 0$ does occur, then $J = 1$ or 2 must be absent. Hence, $J = 3$ must occur nondegenerately and therefore 4 must also occur. Consequently, $J = 5$ must be absent. Thus if $J = 0$ occurs in $(l, 0)$, then $J = 1, 2, 5$ must be missing. On the other hand, if $J = 0$ does not occur in $(l, 0)$, then either $J = 1$ is present or $J = 1$ is missing. If $J = 1$ is missing, one concludes that $J = 3$ is also missing and $J = 2, 4, 5$ each occur without degeneracy.

These results are sufficient to prove the lemma as soon as we have seen that $J = 0$ occurs for $l = 3n$ and not otherwise, and that $J = 1$ never occurs. It is easy to convince oneself of these special results by consideration of degeneracy diagrams.

We are now in a position to prove the main theorem. The method of proof is an adaptation of that used by Elliott[7] in considering the $SU(3)$–$R(3)$ reduction problem and proceeds by *reductio ad absurdum*.

We suppose that the set of functions $\psi(l\nu JM)$ does *not* span the representation space for the irreducible representation $(l, 0)$. It follows that there must exist a function $\varphi(J', M')$ in the representation space of $(l, 0)$ which is orthogonal to all the $\psi(l\nu JM)$. Of course, this would be automatically true if J' were different from any of the J values represented among the functions $\psi(l\nu JM)$, but this possibility is excluded by the lemma. Hence we conclude that there exists a function $\varphi(JM)$ such that the Hilbert space scalar product

$$(\varphi(JM), \psi(l\nu J'M')) = 0$$

for all ν, J', M', and the result is nontrivial only when $J' = J$, $M' = M$.

Hence our hypothesis shows that

$$\begin{aligned}
&\int d\Omega\, D^{J*}_{MK}(\Omega)(\varphi(JM), \Omega\chi(l\nu)) \\
&\quad = \int d\Omega\, D^{J*}_{MK}(\Omega)(\Omega^{-1}\varphi(JM), \chi(l, \nu)) \\
&\quad = \sum_{M'} \int d\Omega\, D^{J*}_{MK}(\Omega)\, D^{J}_{MM'}(\Omega)(\varphi(JM'), \chi(l\nu)) \\
&\quad = (2J+1)^{-1}(\varphi(J, K), \chi(l, \nu)) = 0
\end{aligned} \tag{17}$$

for a suitable choice of the volume element $d\Omega$ in the $R(3)$ parameter space. Hence, since $J_0\chi(l, \nu) = K\chi(l, \nu)$, we must have

$$[\varphi(J, M), \chi(l, \nu)] = 0 \tag{18}$$

for all intrinsic states $\chi(l, \nu)$ and all values of M, not necessarily equal to K.

We now proceed to show that Eq. (18) implies

$$[\varphi(J, M), \mathcal{O}\chi(l, \nu)] = 0, \tag{19}$$

where $\mathcal{O}$ is an arbitrary element of $R(5)$ acting on the intrinsic state $\chi(l, \nu)$. We note that $\mathcal{O}$ can be expressed as a power series in the generators of the group, and that the generators in any particular term may be ordered in any desired manner provided we included a compensating term of lower degree derived from the commutation rules. We divide the generators appropriate to the natural basis into two sets

$$\text{A:}\quad p_{\pm 1},\quad p_0,\quad q_{-1},\quad q_0,\quad T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},-\frac{1}{2}}$$

and

$$\text{B:}\quad q_1,\quad T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\frac{1}{2}}.$$

When a generator of the set A acts on an intrinsic state, the result is either an intrinsic state (if the generator is one of $p_{\pm 1}$, p_0, q_0) or zero (for q_{-1}, $T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},-\frac{1}{2}}$, and if $\nu = 0, p_{+1}$). Generators of set B do not reproduce intrinsic states but are equivalent in their effect to the operators $J_{\pm 1}$ acting on intrinsic states.

To see this, we note from Eqs. (3)–(5) that

$$J_1 = 2q_1 - \sqrt{3}\, T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2},-\frac{1}{2}}$$

and

$$J_{-1} = 2q_{-1} + \sqrt{3}\, T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2},\frac{1}{2}}. \tag{20}$$

From these, together with the fact that $T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2},-\frac{1}{2}}$ and q_{-1} annihilate intrinsic states, it follows that

$$q_1\chi(l,\nu) = \tfrac{1}{2}J_1\chi(l,\nu) \tag{21}$$

and

$$T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2},\frac{1}{2}}\chi(l,\nu) = 3^{-\frac{1}{2}}J_{-1}\chi(l,\nu). \tag{22}$$

The operator $T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2}\frac{1}{2}}$ is more tricky to deal with, and success depends on the fact that for intrinsic states $p = q = \frac{1}{2}l$, i.e., p attains its maximum value. From the explicit matrix elements of $T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\frac{1}{2}}$ which were found in I, specialized to intrinsic states, we find

$$T^{[\frac{1}{2}\frac{1}{2}]}_{\frac{1}{2}\frac{1}{2}}\chi(l,\nu) = T^{[\frac{1}{2}\frac{1}{2}]}_{-\frac{1}{2},\frac{1}{2}}\chi(l,\nu-1) = 3^{-\frac{1}{2}}J_{-1}\chi(l,\nu-1), \tag{23}$$

where the first step utilizes $p = \frac{1}{2}l = p_{\max}$ and the second step follows from Eq. (22).

We now see that

$$(\varphi(J,M), \mathcal{O}\chi(l,\nu)) = \sum_r (\varphi(J,M), \pi_{\mathrm{B}_r}\pi_{\mathrm{A}_r}\chi(l,\nu)), \tag{24}$$

where π_{A_r}, π_{B_r} are products of generators of sets A and B, respectively. Each factor π_{A_r} merely reproduces an intrinsic state [in general, different from $\chi(l,\nu)$], while the factor π_{B_r} is then equivalent to a product of $R(3)$ generators operating on an intrinsic state. However, the $R(3)$ generators may be taken to act on $\varphi(J,M)$, in which case only the M value can be changed. Thus we obtain, finally,

$$\begin{aligned}(\varphi(J,M), \mathcal{O}\chi(l,\nu)) &= \sum_r (\varphi(J,M), \pi_{\mathrm{B}_r}\pi_{\mathrm{A}_r}\chi(l,\nu)) \\ &= \sum_{rM'\nu'} C(r,M',\nu')(\varphi(J,M'),\chi(l,\nu')) = 0 \end{aligned} \tag{25}$$

by Eq. (19).

We are now in a position to complete the *reductio ad absurdum* proof of the theorem. By our hypothesis that the theorem is false, we have found that there exist functions $\varphi(J,M)$ belonging to the representation space for the irreducible representation $(l,0)$ of $R(5)$ which are orthogonal to all the intrinsic states $\chi(l,\nu)$. Then, by Eq. (25), we see that the $\varphi(J,M)$ are orthogonal to all states of the form $\mathcal{O}\chi(l,\nu)$, where $\mathcal{O}$ is an arbitrary element of the group $R(5)$. However, from the irreducibility of the representation $(l,0)$, which implies by definition that the representation space possesses no proper subspace invariant under the group, it follows that from the set of states $\mathcal{O}\chi(l,\nu)$ we can find a subset which spans the complete representation space. Hence, $\varphi(J,M)$ is orthogonal to all states in the representation space of $(l,0)$, which contradicts the hypothesis that $\varphi(J,M)$ belongs to this space. This contradiction is sufficient to prove the theorem.

4. NORMALIZATION AND OVERLAP INTEGRALS

The functions $\psi(l\nu JM)$ defined by Eq. (15) have been shown to be the basis functions for the irreducible representation $(l,0)$ of $R(5)$ fully reduced specifically with respect to the physical $R(3)$. However, if two of these basis functions differ only in the value of ν, they are not orthogonal, nor are any normalized. We shall define the Hilbert-space integral

$$A^l_J(\nu',\nu) \equiv (\psi(l\nu'JM), \psi(l\nu JM)). \tag{26}$$

When $\nu' = \nu$, $A^l_J(\nu,\nu)$ is the square of the normalization constant and we adopt the convention of taking the positive square root. For $\nu' \neq \nu$, Eq. (26) is the overlap integral for states of common J but different ν belonging to the irreducible representation $(l,0)$.

Before we proceed to compute $A^l_J(\nu',\nu)$, let us indicate specifically which J's are involved in the overlap integrals. We shall consider $\nu' \geq \nu$ or, equivalently, $K \geq K'$. From Eq. (12) it follows that $\psi(l\nu'JM)$ and $\psi(l\nu JM)$ have common values of J only for

$$K' = K - 3n. \tag{27}$$

The maximum common J value is $2K'$ and the minimum is K. Therefore,

$$2K' \geq K,$$

which implies that the $A^l_J(\nu',\nu)$ are zero unless

$$\begin{aligned} K' &= K - 3n, \\ n &= 0, 1, 2, \cdots, [K/6]. \end{aligned} \tag{28}$$

The common values of J are the set

$$J = 2K', 2K'-2, 2K'-3, \cdots, K; \tag{29}$$

that is, J runs from K to $2K'$ in steps of 1 excluding $2K'-1$.

In passing, we also note that these considerations will yield an explicit formula for the number of times $N(J)$ that J occurs in the irreducible representation $(l,0)$. Again, with $\nu \leq \nu'$, which implies $K \geq K'$, J occurs for both values of ν provided

$$K \leq J \leq 2K'$$

or

$$l - 3\nu \leq J \leq 2l - 6\nu'.$$

This gives

$$\nu \geq \tfrac{1}{3}(l - J)$$

and

$$\nu' \leq \tfrac{1}{6}(2l - J),$$

or, since ν and ν' are integers,

$$[\tfrac{1}{3}(l - J + 2)] \leq \nu \leq \nu' \leq [\tfrac{1}{6}(2l - J)].$$

Hence, the number of ν values associated with J cannot exceed $[\frac{1}{6}(2l - J)] - [\frac{1}{3}(l - J + 2)] + 1$. This will be $N(J)$ provided we interpret the [ ] as being zero whenever its argument is negative, and unless $2l - J - 1$ is $6n$, where n is an integer, in which case $N(J)$ will be smaller by 1. Hence we conclude that

$$N(J) = \begin{cases} [\frac{1}{6}(2l - J)] - [\frac{1}{3}(l - J + 2)] + 1, \\ \qquad\qquad 2l - J - 1 \neq 6n, \\ [\frac{1}{6}(2l - j)] - [\frac{1}{3}(l - J + 2)], \\ \qquad\qquad 2l - J - 1 = 6n, \end{cases} \tag{30}$$

where n is an integer. Equation (30) is the solution to the induction equations given by Weber *et al.*[12]

From Eq. (26) together with Eq. (15) we have

$$\begin{aligned} A^l_J(\nu', \nu) &= \int d\Omega\, D^J_{MK'}(\Omega)(\chi(l\nu'), \Omega^{-1}\psi(l\nu JM)) \\ &= \sum_{M'} \int d\Omega\, D^J_{MK'}(\Omega)(\chi(l\nu'), \psi(l\nu JM'))D^{J*}_{MM'}(\Omega) \\ &= (2J + 1)^{-1}(\chi(l\nu'), \psi(l\nu JK')). \end{aligned} \tag{31}$$

Again, we use Eq. (15) and find

$$A^l_J(\nu', \nu) = (2J + 1)^{-1} \int d\Omega\, D^{J*}_{K'K}(\Omega)(\chi(l\nu'), \Omega\chi(l\nu)). \tag{32}$$

The problem then is finding $[\chi(l\nu'), \Omega\chi(l\nu)]$. To obtain the required matrix elements, we may construct states with the same transformation properties as $\chi(l\nu)$ in any manner we like. Now $\chi(l\nu)$ is a state in the $R(4) = SU(2) \otimes SU(2)$ space corresponding to

$$(p, q, p_0, q_0) = \left(\frac{l}{2}, \frac{l}{2}, \frac{l}{2} - \nu, -\frac{l}{2}\right).$$

As is well known, the state constructed from functions $\chi_\pm$, corresponding to $(p, q, p_0, q_0) = (\frac{1}{2}, \frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2})$, by

$$\bar{\chi}(l, \nu) = [(l - \nu)!\, \nu!]^{-\frac{1}{2}} \chi_+^{l-\nu} \chi_-^{\nu},$$

has the same transformation properties as $\chi(l\nu)$ and is normalized provided

$$(\chi^a_\alpha, \chi^b_\beta) = \delta_{\alpha\beta}\delta_{ab}a!.$$

But $\chi_\pm$ may be taken as suitable states of the vector representation of $R(5)$, which we know contains only the $J = 2$ representation of $R(3)$. In particular, since $M = 3p_0 + q_0$, we can identify χ_+ with $J = 2$, $M = 1$ and χ_- with $J = 2$, $M = -2$. Similarly, the other states of the vector representation are identified. Clearly $\Omega\chi(l\nu)$ is the same function of $\Omega\chi_+$ and $\Omega\chi_-$ as $\chi(l\nu)$ is of χ_+ and χ_-. But $\Omega\chi_+$ and $\Omega\chi_-$ may be written as linear combinations of χ_+, χ_-, and the other states of the vector representation with appropriate matrix elements of the representation D^2 of $R(3)$ as coefficients. Only the χ_+ and χ_- coefficients need concern us because of the Hilbert-space integral in Eq. (32). Thus, if $\nu' \geq \nu$, we obtain

$$\begin{aligned} &(\chi(l\nu'), \Omega\chi(l\nu)) \\ &= [(l - \nu)!\, \nu!\, (l - \nu')!\, \nu'!]^{\frac{1}{2}} \\ &\quad \times \sum_\beta \frac{(D^2_{11})^{l-\nu-\nu'+\beta}(D^2_{-2,1})^{\nu'-\beta}(D^2_{1,-2})^{\nu-\beta}(D^2_{-2,-2})^\beta}{(l - \nu - \nu' + \beta)!\, (\nu' - \beta)!\, (\nu - \beta)!\, \beta!}, \\ &\qquad\qquad \nu' \geq \nu. \end{aligned} \tag{33}$$

There are several ways to evaluate the integral in Eq. (32) using Eq. (33). Here we merely quote two equivalent forms for $A^l_J(\nu', \nu)$ and relegate the derivations to Appendix B. The first form, suitable for machine computation when one has already available a fast program for Clebsch–Gordan coefficients is

$$\begin{aligned} A^l_J(\nu', \nu) &= (2J + 1)^{-2}[(l - \nu)!\, (l - \nu')!\, \nu!\, \nu'!]^{\frac{1}{2}} \\ &\quad \times \sum_{\beta J' J''} \frac{1}{(l - \nu - \nu' + \beta)!\, (\nu' - \beta)!} \\ &\quad \times \frac{C(l - \nu - \nu' + \beta, J')K(\nu' - \beta)K(\nu - \beta)}{(\nu - \beta)!\, \beta!} \\ &\quad \times C(2\nu - 2\beta, 2\beta, 2\nu'; \nu - \beta, -2\beta) \\ &\quad \times C(2\nu' - 2\beta, 2\nu, J''; -2\nu' + 2\beta, \nu - 3\beta) \\ &\quad \times C(2\nu' - 2\beta, 2\nu, J''; \nu' - \beta, -2\nu) \\ &\quad \times C(J'J''J; l - \nu - \nu' + \beta, \nu - 2\nu' - \beta) \\ &\quad \times C(J'J''J; l - \nu - \nu' + \beta, \nu' - 2\nu - \beta), \\ &\qquad\qquad \nu' \geq \nu, \end{aligned} \tag{34}$$

where

$$K(x) = 2^x \left[\frac{(3x)!\, x!}{(4x)!}\right]^{\frac{1}{2}}, \tag{35}$$

and

$$\begin{aligned} C(l, J) &= \frac{4^{J-l} l!\, (J + l)!}{(2l - J)!(2J)!} \\ &\quad \times {}_2F_1(J - 2l, J - l + 1; 2J + 2; 4). \end{aligned} \tag{36}$$

Alternatively, the $C(l, J)$ may be given by the recursion relationship

$$C(l, J) = \sum_{J'} C(l - 1, J')[C(J'2J; l - 1, 1)]^2$$

with

$$C(0, J) = \delta_{J,0}. \tag{37}$$

The J values belonging to $C(l, J)$ are those values of J in the $(l, 0)$ representation with $K = l$ [Eq. (13)].

[12] H. J. Weber, M. G. Haker, and W. Greiner, Z. Physik **190**, 25 1 966).

Our second form for $A^l_J(\nu', \nu)$ is

$$A^l_J(\nu', \nu) = \frac{2^{\nu'-\nu}}{(2J+1)}\left[\frac{(l-\nu)!\,(l-\nu')!\,\nu!\,\nu'!\,(J-K)!\,(J-K')!}{(J+K)!\,(J+K')!}\right]^{\frac{1}{2}}$$
$$\times \sum_{\alpha\beta\gamma} \frac{4^{\alpha}(-1)^{\alpha+\gamma}(J+K+\gamma)!}{(l-\nu'-\alpha)!\,(\alpha-\beta)!\,(\nu'-\nu+\beta)!\,\beta!\,(\nu-\beta)!}$$
$$\times \frac{(2l-2\nu'-2\beta)!\,(3\nu'-3\nu+2\beta+\alpha+\gamma)!}{(J-K-\gamma)!\,(K-K'+\gamma)!\,(2l+\nu'-3\nu+\alpha+\gamma+1)!}, \quad \nu' \geq \nu. \tag{38}$$

At first sight, this would appear to be no simpler than Eq. (34). However, when it is remembered that the $C(l, J)$ and each Clebsch–Gordan coefficient in Eq. (34) is itself a sum, Eq. (38) then appears to be a considerable simplification. If ν' is smaller than ν, Eqs. (34) and (38) still apply, but with $\nu' \leftrightarrow \nu$ and an additional factor $(-1)^{\nu-\nu'}$. This is equivalent to the symmetry rule

$$A^l_J(\nu', \nu) = (-1)^{\nu-\nu'}A^l_J(\nu, \nu'). \tag{39}$$

5. REDUCED MATRIX ELEMENTS OF $R(5)$ GENERATORS IN THE PHYSICAL BASIS

So far, we have managed to define a complete set of states $\psi(l\nu JM)$, where J is the physical angular momentum. It remains to obtain explicit matrix elements between such states for all generators of $R(5)$. This is trivial for those generators, which are also the generators of $R(3)$. We have

$$\langle\psi(l\nu' JM')|\, J_\lambda \,|\psi(l\nu JM)\rangle = A^l_J(\nu', \nu)[J(J+1)]^{\frac{1}{2}}C(J1J;\, M\lambda M'). \tag{40}$$

The remaining generators are components $Q^{[3]}_\lambda$ of a third rank tensor with respect to $R(3)$, and their matrix elements may be written

$$\langle\psi(l\nu' J'M')|\, Q^{[3]}_\lambda \,|\psi(l\nu JM)\rangle = C(J3J';\, M\lambda M')\,\langle l\nu' J' \|\, Q \,\| l\nu J\rangle, \tag{41}$$

where the sole remaining problem is the evaluation of the reduced matrix element $\langle l\nu' J'\|\, Q \,\|l\nu J\rangle$, which we obtain by considering the particular component $Q^{[3]}_0$ or, more briefly, Q_0. Then

$$Q_0\psi(l\nu JM) = \int D^{J*}_{MK}(\Omega)Q_0\chi_\Omega(l, \nu)\, d\Omega. \tag{42}$$

We make use of the explicit properties of the $Q^{[3]}_\mu$ under $R(3)$ and rewrite Eq. (42) as

$$Q_0\psi(l\nu JM) = \sum_\rho \int D^{J*}_{MK}(\Omega)D^{3*}_{0\rho}(\Omega)Q_\rho(\Omega)\chi_\Omega(l, \nu)\, d\Omega$$
$$= \sum_{\rho J'} C(J3J';\, M0)C(J3J';\, K\rho) \times \int D^{J'*}_{M,K+\rho}(\Omega)Q_\rho(\Omega)\chi_\Omega(l, \nu)\, d\Omega. \tag{43}$$

This has reduced the problem to finding the effect of $Q_\rho(\Omega)$ upon $\chi_\Omega(l, \nu)$ or, what amounts to the same thing, the effect of Q_ρ upon $\chi(l, \nu)$. The Q_ρ may be expressed in terms of the natural basis generators, using Eqs. (3)–(5), as

$$\begin{aligned} Q_{\pm 3} &= 10^{\frac{1}{2}}p_{\pm 1}, \\ Q_{\pm 2} &= \pm 5^{\frac{1}{2}}T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\pm\frac{1}{2}}, \\ Q_{\pm 1} &= 6^{\frac{1}{2}}q_{\pm 1} \pm 2^{\frac{1}{2}}T^{[\frac{1}{2}\frac{1}{2}]}_{\pm\frac{1}{2},\mp\frac{1}{2}}, \end{aligned} \tag{44}$$

and

$$Q_0 = 3q_0 - p_0.$$

Then, operating upon the intrinsic states and using the results of I, we find

$$\begin{aligned} Q_3\chi(l, \nu) &= -[5\nu(l-\nu+1)]^{\frac{1}{2}}\chi(l, \nu-1), \\ Q_{-3}\chi(l, \nu) &= [5(l-\nu)(\nu+1)]^{\frac{1}{2}}\chi(l, \nu+1), \\ Q_{-2}\chi(l, \nu) &= 0, \\ Q_0\chi(l, \nu) &= -(2l-\nu)\chi(l, \nu). \end{aligned} \tag{45}$$

For the remaining components, we have

$$\begin{aligned} Q_2\chi(l, \nu) &= [\tfrac{5}{3}]^{\frac{1}{2}}J_{-1}\chi(l, \nu-1), \\ Q_1\chi(l, \nu) &= \tfrac{1}{2}6^{\frac{1}{2}}J_1\chi(l, \nu), \\ Q_{-1}\chi(l, \nu) &= -[\tfrac{2}{3}]^{\frac{1}{2}}J_{-1}\chi(l, \nu). \end{aligned} \tag{46}$$

Now, from Eq. (15),

$$\chi(l, \nu) = \sum_J (2J+1)\psi(l\nu JK)$$

follows as a trivial corollary. We shall use this to evaluate

$$\mathfrak{J}_\pm \equiv \int D^{J'*}_{M,K'}(\Omega)J_{\pm 1}(\Omega)\chi_\Omega(l, \nu)\, d\Omega. \tag{47}$$

Then from Eq. (15) we have

$$\begin{aligned} &J_{\pm 1}(\Omega)\chi_\Omega(l, \nu) \\ &= \sum_J (2J+1)J_{\pm 1}(\Omega)\psi_\Omega(l\nu JK) \\ &= \sum_J (2J+1)[J(J+1)]^{\frac{1}{2}}C(J1J;\, K, \pm 1)\psi_\Omega(l\nu JK\pm 1) \\ &= \sum_{J\rho} (2J+1)[J(J+1)]^{\frac{1}{2}}C(J1J;\, K, \pm 1)D^J_{\rho,K\pm 1}(\Omega) \\ &\qquad \times \psi(l\nu J\rho). \end{aligned} \tag{48}$$

Thus,

$$\mathfrak{J}_{\pm} = [J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K' \mp 1, \pm 1, K)\psi(l\nu J'M). \tag{49}$$

Finally then, we find

$$\begin{aligned}
&Q_0\psi(l\nu JM)\\
&= \sum_{J'} C(J3J'; M, 0)\{-C(J3J'; K, 3)[5\nu(l-\nu+1)]^{\frac{1}{2}}\\
&\quad \times \psi(l, \nu-1, J', M) + C(J3J'; K, 2)\\
&\quad \times [\tfrac{5}{3}J'(J+1)]^{\frac{1}{2}} C(J'1J'; K+3, -1)\\
&\quad \times \psi(l, \nu-1, J', M) + C(J3J'; K, 1)\\
&\quad \times [\tfrac{3}{2}J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K, 1)\, \psi(l, \nu, J', M)\\
&\quad - (2l-\nu) C(J3J'; K, 0)\psi(l, \nu, J', M)\\
&\quad - [\tfrac{5}{3}J'(J'+1)]^{\frac{1}{2}} C(J3J'; K, -1) C(J'1J'; K, -1)\\
&\quad \times \psi(l, \nu, J', M) + [5(l-\nu)(\nu+1)]^{\frac{1}{2}}\\
&\quad \times C(J3J'; K, -3)\psi(l, \nu+1, J', M)\}.
\end{aligned} \tag{50}$$

In interpreting Eq. (50), we note that if $\nu = [l/3]$, then $\chi(l, \nu+1)$ is not an intrinsic state as we have previously defined it. In this case we continue to define $\psi(l, \nu+1, J, M)$ by the projection equation (15) from $\chi(l, \nu+1)$ and note that the evaluation of the overlap integrals in Appendix B remains valid. From Eqs. (50) and (41) we now see that

$$\begin{aligned}
&\langle l\nu'J' \| Q \| l\nu J\rangle\\
&= \{-[5\nu(l-\nu+1)]^{\frac{1}{2}} C(J3J'; K, 3)\\
&\quad + [\tfrac{5}{3}J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K+3, -1) C(J3J'; K, 2)\}\\
&\quad \times A^l_{J'}(\nu', \nu-1)\\
&\quad + [5(l-\nu)(\nu+1)]^{\frac{1}{2}} C(J3J'; K, -3) A^l_{J'}(\nu', \nu+1)\\
&\quad + \{[\tfrac{3}{2}J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K, 1) C(J3J'; K, 1)\\
&\quad - [\tfrac{5}{3}J'(J'+1)]^{\frac{1}{2}} C(J'1J'; K, -1) C(J3J'; K, -1)\\
&\quad - (2l-\nu) C(J3J'; K, 0)\} A^l_{J'}(\nu', \nu).
\end{aligned} \tag{51}$$

APPENDIX A

In this appendix we want to show that the rules given by Eqs. (11)–(13) produce the dimension formula

$$d(l) = \tfrac{1}{6}(l+1)(l+2)(2l+3). \tag{A1}$$

Now, according to Eq. (13) for given K, the total number of states is

$$d_K = \sum_{J=K}^{2K}{}'(2J+1), \tag{A2}$$

where we have specifically indicated by the prime that the value $2K-1$ is excluded from the sum. Therefore,

$$\begin{aligned}
d_K &= (4K+1) + \sum_{J=K}^{2K-2}(2J+1)\\
&= 4K + 1 + (3K-1)(K-1)\\
&= 3K^2 + 2.
\end{aligned} \tag{A3}$$

This is clearly valid so long as $K \neq 0$. For $K = 0$ we have $d_0 = 1$. Thus for all K we have

$$d_K = 3K^2 + 2 - \delta_{K,0}. \tag{A4}$$

Now let us write

$$l = 3n + m, \quad n = 0, 1, 2 \cdots, \quad m = 0, 1, 2. \tag{A5}$$

Then with $K = l - 3\nu$ we have $\nu = 0, 1, 2, \cdots, n$. Thus, in terms of ν

$$d_\nu = 3(l-3\nu)^2 + 2 - \delta_{m,0}\delta_{\nu,n}. \tag{A6}$$

The dimension of $(l, 0)$ is therefore given by

$$\begin{aligned}
d(l) = \sum_{\nu=0}^{n} d_\nu &= (3l^2+2)(n+1) - \delta_{m,0}\\
&\quad - 9ln(n+1) + \tfrac{9}{2}n(n+1)(2n+1).
\end{aligned} \tag{A7}$$

We use $n = (l-m)/3$ and after some rearrangement find

$$\begin{aligned}
d(l) = \tfrac{1}{6}\{&(l+1)(l+2)(2l+3)\\
&- (m-1)(m-2)(2m-3) - 6\delta_{m,0}\}.
\end{aligned} \tag{A8}$$

But since $m = 0$, 1, or 2 only, we find the desired result.

APPENDIX B

We want to evaluate the quantity given in Eq. (32) for $\nu' \geq \nu$:

$$\begin{aligned}
&A^l_J(\nu', \nu)\\
&= (2J+1)^{-1}\int d\Omega\, D^{J*}_{K'K}(\Omega)(X(l, \nu'), \Omega X(l, \nu)),
\end{aligned} \tag{B1}$$

where

$$(X(l,\nu'), \Omega X(l,\nu)) = [(l-\nu)!\,\nu!\,(l-\nu')!\,\nu'!]^{\frac{1}{2}} \sum_\beta \frac{[D^2_{11}(\Omega)]^{l-\nu-\nu'+\beta}[D^2_{-2,1}(\Omega)]^{\nu'-\beta}[D^2_{1,-2}(\Omega)]^{\nu-\beta}[D^2_{-2,-2}(\Omega)]^{\beta}}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!}. \tag{B2}$$

Now,

$$[D^2_{11}(\Omega)]^\alpha \equiv \sum_{J'} C(\alpha, J') D^{J'}_{\alpha\alpha}(\Omega), \tag{B3}$$

where the J' values in the sum are those from α to 2α in steps of 1 with the exception of $2\alpha - 1$, i.e., those in $(\alpha, 0)$ with $K = \alpha$. Also

$$[D^2_{-2,1}(\Omega)]^\beta \equiv K(\beta) D^{2\beta}_{-2\beta,\beta}(\Omega), \tag{B4}$$

and then clearly,

$$[D^2_{1,-2}(\Omega)]^\gamma = K(\gamma) D^{2\gamma}_{\gamma,-2\gamma}(\Omega);$$

finally,

$$[D^2_{-2,-2}(\Omega)]^\delta = D^{2\delta}_{-2\delta,-2\delta}(\Omega). \quad \text{(B5)}$$

Then we have

$$\begin{aligned}&(X(l,\nu'),\Omega X(l\nu))\\ &= [(l-\nu)!\,\nu!\,(l-\nu')!\,\nu'!]^{\frac12}\\ &\times \sum_{\beta J'} \frac{C(l-\nu-\nu'+\beta,J')K(\nu'-\beta)K(\nu-\beta)}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!}\\ &\times D^{J'}_{l-\nu-\nu'+\beta,l-\nu-\nu'+\beta}(\Omega)\,D^{2(\nu'-\beta)}_{-2(\nu'-\beta),\nu'-\beta}(\Omega)\\ &\times D^{2(\nu-\beta)}_{\nu-\beta,-2(\nu-\beta)}(\Omega)\,D^{2\beta}_{-2\beta,-2\beta}(\Omega). \quad \text{(B6)}\end{aligned}$$

In Eq. (B6), we couple the last two rotation matrices to 2ν, couple the result with $D^{2(\nu'-\beta)}(\Omega)$, and, finally, couple the result to $D^{J'}(\Omega)$. We find

$$\begin{aligned}&(X(l\nu'),\Omega X(l\nu))\\ &= [(l-\nu)!\,\nu!\,(l-\nu')!\,\nu'!]^{\frac12}\\ &\times \sum_{\beta J'J''J'''} \frac{C(l-\nu-\nu'+\beta,J')K(\nu'-\beta)K(\nu-\beta)}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!}\\ &\times C(2(\nu-\beta)2\beta2\nu;\nu-\beta,-2\beta)\\ &\times C(2(\nu'-\beta)2\nu J'';-2\nu'+2\beta,\nu-3\beta)\\ &\times C(2(\nu'-\beta)2\nu J'';\nu'-\beta,-2\nu)\\ &\times C(J'J''J''';l-\nu-\nu'+\beta,\nu-2\nu'-\beta)\\ &\times C(J'J''J''';l-\nu-\nu'+\beta,\nu'-2\nu-\beta)\\ &\times D^{J'''}_{K',K}(\Omega). \quad \text{(B7)}\end{aligned}$$

When Eq. (B7) is inserted into Eq. (B1) and the resulting integration performed, we find the result given in Eq. (34).

Now, the quantity $K(\beta)$ is defined by Eq. (B4). Thus we see that $K(0)=1$. From

$$(D^2_{-2,1}(\Omega))^\beta = (D^2_{-2,1}(\Omega))^{\beta-1}D^2_{-2,1}(\Omega)$$

it follows that

$$\begin{aligned}K(\beta) &= K(\beta-1)C(2(\beta-1),2,2\beta;\beta-1,1)\\ &= K(\beta-1)\left[\frac{(3\beta-2)(3\beta-1)(3\beta)}{(4\beta-3)(4\beta-2)(4\beta-1)}\right]^{\frac12}. \quad \text{(B8)}\end{aligned}$$

This induction equation can clearly be iterated to yield

$$K(\beta) = 2^x\left[\frac{(3\beta)!\,[4(\beta-x)]!\,\beta!}{[3(\beta-x)]!\,(4\beta)!\,(\beta-x)!}\right]^{\frac12} K(\beta-x) \quad \text{(B9)}$$

and, upon setting $x=\beta$ in Eq. (B9), we find the result given in Eq. (35).

The quantities $C(\alpha,J)$ are defined by Eq. (B3). From the orthogonality of the irreducible representation of $R(3)$ it follows that

$$C(\alpha,J) = (2J+1)\int d\Omega\, D^{J*}_{\alpha\alpha}(\Omega)(D^2_{11}(\Omega))^\alpha. \quad \text{(B10)}$$

We use

$$D^J_{\alpha\beta}(\Omega) = e^{-i\alpha\theta_1}d^J_{\alpha\beta}(\theta_2)e^{-i\gamma\theta_3},$$

where the θ_i are Euler angles, and then we rewrite Eq. (B10) as

$$C(\alpha,J) = \frac{2J+1}{2}\int_0^\pi \sin\theta\, d\theta\, d^J_{\alpha\alpha}(\theta)(d^2_{11}(\theta))^\alpha. \quad \text{(B11)}$$

The $d^J_{\alpha\alpha}(\theta)$ are given by[10]

$$\begin{aligned}d^J_{\alpha\beta}(\theta) &= \frac{1}{(\alpha-\beta)!}\left[\frac{(J-\beta)!\,(J+\alpha)!}{(J-\alpha)!\,(J+\beta)!}\right]^{\frac12}\\ &\times(\cos\theta/2)^{2J+\beta-\alpha}(-\sin\theta/2)^{\alpha-\beta}\\ &\times {}_2F_1(\alpha-J,-\beta-J;\alpha-\beta+1;-\tan^2\theta/2),\\ &\qquad \alpha\ge\beta. \quad \text{(B12)}\end{aligned}$$

We use Eq. (B12) in Eq. (B11) after changing the variable to $z=\frac12(1-\cos\theta)$ and using the transformation of the ${}_2F_1$ function

$${}_2F_1(a,b;c;x) = (1-x)^{-a}{}_2F_1\left(a,c-b;c;\frac{x}{x-1}\right) \quad \text{(B13)}$$

to find

$$\begin{aligned}C(\alpha,J) &= (2J+1)\int_0^1(1-z)^{2\alpha}(1-4z)^\alpha\\ &\times {}_2F_1(\alpha-J,\alpha+J+1;1;z)\,dz. \quad \text{(B14)}\end{aligned}$$

Now

$$\begin{aligned}&{}_2F_1(\alpha-J,\alpha+J+1;1;z)\\ &= \frac{1}{(J-\alpha)!}(1-z)^{-2\alpha}\frac{d^{J-\alpha}}{d^{J-\alpha}}[z^{J-\alpha}(1-z)^{J+\alpha}]\end{aligned}$$

so we may do $J-\alpha$ partial integrations in Eq. (B14) and the integrated parts vanish. Thus

$$\begin{aligned}C(\alpha,J) &= \frac{(2J+1)}{(J-\alpha)!}4^{J-\alpha}\frac{\alpha!}{(2\alpha-J)!}\\ &\times\int_0^1(1-4z)^{2\alpha-J}z^{J-\alpha}(1-z)^{J+\alpha}\,dz. \quad \text{(B15)}\end{aligned}$$

The integral in Eq. (B15) is a standard integral representation[13] for a hypergeometric ${}_2F_1$ function and so we have the result quoted in Eq. (36). We note that $C(\alpha,J)$ is specifically zero for $J<\alpha$, $J>2\alpha$, or $J=2\alpha-1$. The recursion relationship for $C(\alpha,J)$ follows trivially from the definition Eq. (B3).

We shall now derive the second form for $A^l_J(\nu',\nu)$. Again we use Eq. (B2) in Eq. (B1), but now we

[13] M. Abramowitz and I. A. Stegum, *Handbook of Mathematical Functions* (National Bureau of Standards, AMS. 55, 1964).

specifically perform the azimuthal angle integrations to find

$$A^l_J(\nu',\nu) = \frac{1}{2(2J+1)}[(l-\nu)!\,(l-\nu')!\,\nu!\,\nu'!]^{\frac{1}{2}} \times \sum_\beta \frac{1}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!} \frac{(-1)^{\nu-\beta}}{(\nu-\beta)!\,\beta!} \times \int_0^\pi \sin\theta\, d\theta d^J_{K'K}(\theta)[d^2_{11}(\theta)]^{l-\nu-\nu'+\beta} \times [d^2_{-2,1}(\theta)]^{\nu'+\nu-2\beta}[d^2_{-2,-2}(\theta)]^\beta. \quad \text{(B16)}$$

Again, we use Eq. (B12), wherein we must take care that the left subscript is the larger; if it is not, we must use

$$d^J_{\alpha\beta}(\theta) = d^J_{\beta\alpha}(-\theta).$$

At the same time we shall also use Eq. (B13) and change the variable to $z = \frac{1}{2}(1-\cos\theta)$. Then we find

$$A^l_J(\nu',\nu) = \frac{2^{\nu'-\nu}}{(2J+1)(K-K')!} \times \left[\frac{(l-\nu)!(l-\nu')!\nu!\nu'!(J-K')!(J+K)!}{(J+K')!\,(J-K)!}\right]^{\frac{1}{2}} \times \sum_\beta \frac{(-4)^{\nu-\beta}}{(l-\nu-\nu'+\beta)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!} \times \int_0^1 dz(1-z)^{2(l-\nu-\nu'+\beta)}z^{3(\nu'-\beta)}(1-4z)^{l-\nu-\nu'+\beta} \times {}_2F_1(K-J, K+J+1; K-K'+1; z). \quad \text{(B17)}$$

We expand $(1-4z)^{l-\nu-\nu'+\beta}$ in the integral and find

$$A^l_J(\nu',\nu) = \frac{2^{\nu'-\nu}}{(2J+1)(K-K')!} \times \left[\frac{(l-\nu)!(l-\nu')!\,\nu!\,\nu'!\,(J-K')!\,(J+K)!}{(J+K')!\,(J-K)!}\right]^{\frac{1}{2}} \times \sum_{\alpha\beta} \frac{(-4)^{\nu-\beta+\alpha}}{(l-\nu-\nu'+\beta-\alpha)!\,(\nu'-\beta)!\,(\nu-\beta)!\,\beta!\,\alpha!} \times \int_0^1 dz(1-z)^{2(l-\nu-\nu'+\beta)}z^{3\nu'-3\beta+\alpha} \times {}_2F_1(K-J, K+J+1; K-K'+1; z). \quad \text{(B18)}$$

But, from Eqs. B5.5.2(6) and B5.6(1) of the Bateman[14] papers, it follows that

$$\int_0^1 y^{b-1}(1-y)^{c-b-1}{}_pF_q(a_1,\cdots,a_p; c_1,\cdots,c_q; xy) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}{}_{p+1}F_{q+1}(a_1,\cdots,a_p,b; c_1,\cdots,c_q,c; x), \quad \text{(B19)}$$

provided $q < p+1$, $|\arg x| < \pi$, $\operatorname{Re} c > \operatorname{Re} b > 0$. Thus we find

$$A^l_J(\nu',\nu) = \frac{2^{\nu'-\nu}}{(2J+1)(K-K')!} \times \left[\frac{(l-\nu)!\,(l-\nu')!\,\nu!\,\nu'!\,(J-K')!\,(J+K)!}{(J+K')!\,(J-K)!}\right]^{\frac{1}{2}} \times \sum_{\alpha\beta} \frac{(-4)^{\nu-\beta+\alpha}(3\nu'-3\beta+\alpha)!}{(l-\nu-\nu'+\beta-\alpha)!\,(\nu'-\beta)!\,(\nu-\beta)!} \times \frac{(2l-2\nu-2\nu'+2\beta)!}{\beta!\,\alpha!\,(2l+\nu'-2\nu-\beta+\alpha+1)!} \times {}_3F_2(K-J, K+J+1, 3\nu'-3\beta+\alpha+1; K-K'+1, 2l+\nu'-2\nu-\beta+\alpha+2; 1). \quad \text{(B20)}$$

Unfortunately, the ${}_3F_2$ is neither well poised nor Saalschützian, so the final result is a triple sum:

$$A^l_J(\nu',\nu) = \frac{2^{\nu'-\nu}}{(2J+1)}\left[\frac{(l-\nu)!\,(l-\nu')!\,\nu!\,\nu'!(J-K)!\,(J-K')!}{(J+K)!\,(J+K')!}\right] \times \sum_{\alpha\beta\gamma} \frac{4^\alpha(-1)^{\alpha+\gamma}(J+K+\gamma)!\,(2l-2\nu'-2\beta)!}{(l-\nu'-\alpha)!\,(\alpha-\beta)!\,(\nu'-\nu+\beta)!\,\beta!\,(\nu-\beta)!} \times \frac{(3\nu'-3\nu+2\beta+\alpha+\gamma)!}{(J-K-\gamma)!\,(K-K'+\gamma)!\,(2l+\nu'-3\nu+\alpha+\gamma+1)!}. \quad \text{(B21)}$$

[14] A. Erdelyi *et al.*, *Bateman Manuscript Project, Higher Transcendental Functions* (McGraw-Hill Book Co., Inc., New York, 1953), Vol. I. We shall denote equations of this reference by section number, e.g., B5.6 and equation number therein, e.g., (1).

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Approach to Scattering Problems through Interpolation Formulas and Application to Spin-Orbit Potentials*

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In many problems of potential scattering, and particularly in the inverse-scattering problem, two equations prove to be of essential interest: the Gel'fand-Levitan equation and the Regge-Newton equation. These and their generalizations apply, respectively, to energy-independent and to λ-independent potentials. In this paper, a method is devised for obtaining equations applying to more general cases. The requirement is the existence of analytic properties of the wavefunctions corresponding to special classes of potentials, enabling one to construct interpolation formulas of the Lagrange form. From these formulas, it is possible to derive integral equations which may then be generalizable to much larger classes of potentials. This method is fully developed in the case of potentials depending linearly on λ. Interpolation formulas and analytic properties of the wavefunctions in the λ plane are exhibited. Integral equations are given and proved to apply to very large classes of potentials. Existence, uniqueness, and analytic properties of their solutions are thoroughly studied. An example is given for which all the wavefunctions are calculated exactly. Application of the method to the inverse scattering problem in the presence of a spin-orbit potential will be the object of a forthcoming paper.

1. INTRODUCTION

Among the most powerful tools for dealing with problems of scattering by a central potential are the Gel'fand-Levitan equation[1] and its analog, the Regge-Newton equation.[2,3] Quantities of interest, like wavefunctions, Jost functions, phase shifts, etc., may depend on three parameters—the energy E, the angular momentum λ $(= l + \frac{1}{2})$, and the reduced distance r.[4] The Gel'fand-Levitan equation is useful when λ is fixed, while E and r are treated as real or complex parameters. The Regge-Newton equation is useful when E is fixed, while λ and r are real or complex parameters. Although the primary interest of these equations has been the inverse-scattering problem (at fixed l and at fixed energy, respectively), they have also proved useful for deriving analytical properties of the wavefunctions and of the Jost functions, respectively, in the E plane[5] and in the λ plane.[6] It is therefore interesting to devise methods enabling one to derive generalizations of these integral equations applying to scattering problems in which the known forms do not fit.[7] Examples of such problems are, at fixed l, scattering by energy-dependent potentials and, at fixed energy, scattering by λ-dependent potentials. The latter case is certainly the more interesting, since it involves physical problems such as scattering of a spinning particle by a spin-dependent potential. We shall study it in the present paper. The principles of our method are given in the Introduction, and the method is fully developed through treating the scattering by a spin-orbit potential in the remaining two sections of the paper. The starting point in the method is the following remark: *For central potentials, the special analytic properties of the wavefunctions corresponding to special subclasses of potentials enable us to write down interpolation formulas which are trivially equivalent to the Regge-Newton equation and the related integral formulas.*

Let us sketch the argument. Let $\psi_\lambda(z)$ and $\chi_\lambda(z)$ be two regular wavefunctions corresponding to the angular momentum λ and to potentials which are both even analytic functions of z in a circle centered

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1 I. M. Gel'fand and B. M. Levitan, Isv. Akad. Nauk. SSSR **15**, 309 (1959). Translated in Am. Math. Soc. Transl. **1**, 253 (1955). See also L. D. Faddeyev, J. Math. Phys. **4**, 72 (1963).

2 This equation has been introduced by T. Regge [Nuovo Cimento **14**, 951 (1949)] in order to show the uniqueness of a potential giving rise to a partial wave amplitude having a given dynamical interpolation in the angular-momentum plane.

3 R. G. Newton [J. Math. Phys. 3, 75 (1962)] was the first to apply this equation to the construction of potentials from the phase shifts at fixed energy. Studies of the analytic and asymptotic properties of the solutions, generalizations and further applications to the inverse scattering problem, generalization to any base and connection with interpolation properties of the wavefunctions have been given recently by the author [J. Math. Phys. 7, 1515 (1966) and Refs. 6, 11, and 12 below].

4 This reduced distance is measured hereafter in units of k_0^{-1}, the reduced wavelength of the relative motion at a standard positive energy, which is fixed throughout the following. We use for r the notation z when it is meant as a complex parameter.

5 See, for example, the recent papers of H. Cornille on N/D equations [preprints of Laboratoire de Physique Nucléaire Orsay]; J. Math. Phys. (to be published).

6 P. C. Sabatier, J. Math. Phys. 7, 2079 (1966); see also Refs. 11 and 12 below.

7 A list of references concerning generalizations of the Gel'fand-Levitan-Marchenko procedure is given in a book by R. G. Newton, *Quantum Scattering Theory* (McGraw-Hill Book Company, New York, 1966), p. 632.

at the origin. Then we know[8] that the function

$$\lambda g(\lambda) = \lambda \chi_\lambda(z) \frac{\psi_{-\lambda}(z)\psi'_\mu(z) - \psi'_{-\lambda}(z)\psi_\mu(z)}{\mu + \lambda} \quad (1.1)$$

is an entire function of λ of order 1, type π, bounded on the real axis, going to zero[9] like $C\lambda$ as λ goes to zero. As a result, as we see below (Sec. 2E), we can write down for $g(\lambda)$ an interpolation formula of the Lagrange form, in which the dependence on λ is given explicitly in terms of the values of $g(\lambda)$ at integers. Now, since the values of wavefunctions for opposite integer λ's are proportional to each other, it is possible to write down a formula including only positive integers and the proportionality coefficients ψ_{-n}/ψ_n and χ_{-n}/χ_n, which we denote, respectively, by $\gamma_n(\psi)$ and $\gamma_n(\chi)$. Some elementary algebra, and making μ equal to λ, leads one to the fundamental formula

$$\chi_\lambda(z) = \psi_\lambda(z) + \sum_n \frac{\psi_n(z)\psi'_\lambda(z) - \psi'_n(z)\psi_\lambda(z)}{\lambda^2 - n^2} [2\pi^{-1}n(\gamma_n(\psi) - \gamma_n(\chi))]\chi_n(z) \quad (1.2)$$

or

$$\chi_\lambda(z) = \psi_\lambda(z) - \int_0^z \psi_\lambda(\rho)\rho^{-2}\, d\rho \times \sum_n \gamma_n(W_0, V_0)\psi_n(\rho)\chi_n(z), \quad (1.3)$$

where we have used the notation $\gamma_n(W_0, V_0)$ for $2\pi^{-1}n[\gamma_n(\psi) - \gamma_n(\chi)]$, W_0 and V_0 being, respectively, the potentials corresponding to χ and ψ. The series on the right-hand side of (1.3) can be used to define a function of two variables $K(z, \rho)$, so that (1.3) is nothing but the integral formula for the wavefunction. Multiplying both sides by $\gamma_\lambda(W_0, V_0)\psi_\lambda(z')$ and summing over all the positive integral values of λ yield the Regge–Newton equation.

Considerations of the same kind enable us to derive the Regge–Newton equation for potentials which are analytic functions of any rational power of r. The only change in the method is that, in order to obtain an entire function, we have to multiply the function (1.1) by $\sin 2m\pi\lambda/2m \sin \pi\lambda$, where m is a conveniently chosen integer. The function then obtained is of type $2m\pi$, and the sequence involved therefore includes numbers of the form $n/2m$, where n is an integer. This modification is the only one in (1.2) and (1.3).

The preceding remark, applying to central λ-independent potentials, suggests a way of obtaining integral equations which apply to more general problems. First, we notice that, in order to ensure the analytic properties of (1.1), it is sufficient that

$$\frac{\psi_\lambda(z)}{\Gamma(-\lambda)} = e^{\lambda g(z)}\psi(\lambda, z), \quad (1.4)$$

where $\psi(\lambda, z)$ and $(\partial/\partial z)[\psi(\lambda, z)]$ are entire functions of λ of order 1, type π, bounded on the real axis, with z belonging to some continuous domain, and $\psi_\lambda(z)$ is itself an entire function. Generalizations to the case when $\psi_\lambda(z)$ has poles on periodic sequences of negative λ's can easily be done as above—i.e., by multiplying by appropriate functions. Changes of normalization of $\psi_\lambda(z)$ can also be made, with the use of some care.

The second way of extending the method rests on the following remark: we do not need ψ_{-n} to be related precisely to ψ_n. We can content ourselves with cross relations between the "components" $\psi_\lambda^{(i)}$ of a kind of vectorial solution of a sequence *of coupled or uncoupled* Schrödinger equations. In the case which we study thoroughly in the present paper, these cross relations are encountered between the solutions of otherwise uncoupled Schrödinger equations corresponding to two spin states. This remark paves the way for a study of Schrödinger equations coupled by a matricial potential.

The interesting point in this method is that it starts with special properties obtained for special classes of potentials. It is needless to say that, once the generalization of the integral equation has been given, we still have to show that it works for much larger classes. However, this is much easier than having also to look for its form. Furthermore, just as the formal analogy between the study of the Schrödinger equation at fixed l, variable E, and the one at fixed E, variable λ, suggested the Regge–Newton equation from the Gel'fand–Levitan equation, so we can hope to obtain, in a reverse way, integral equations useful at fixed l from integral equations obtained for fixed E. It is probably possible, but more intricate, to construct a method for obtaining these equations from generalizations of the interpolation formulas[10] in the E plane.

In the present paper, the method is fully developed for the example of potentials depending linearly on λ. This example corresponds in physics to the scattering of a spin-$\frac{1}{2}$ particle by a spin-orbit potential ($\mathbf{L} \cdot \mathbf{S}$). It is remarkable that the two wave equations thus obtained, although uncoupled as regards their

[8] This property follows readily from the analytical properties of $\psi_\lambda(z)$ given by the author in a recent paper [Compt. Rend. Acad. Sci. A **263**, 788 (1967)]; see also Ref. 12 below. The proof given in Sec. 2E of the present paper applies also to (1.1), by making the spin–orbit potential equal to zero.

[9] Throughout this paper, we use C as a general constant, and we mean by ϵ a positive number which can be made arbitrarily small. Both C and ϵ are not meant to have the same value everytime they are used.

[10] These formulas, given by K. Chadan [Nuovo Cimento **39**, 697 (1965)] have a form much less convenient than their analog in the λ plane.

physical solutions, are coupled as regards the integral equations system. From the point of view of our method, the significantly original point in this example is that the analytic properties of (1.1) are themselves derived from a simpler interpolation formula, which can be obtained from the Schrödinger equation in elementary ways. The obtaining of all the analytic properties and interpolation formulas is the concern of Sec. 2. In Sec. 3 we give the integral equations, prove them for larger classes of potentials, give the integral formulas, and reduce the integral equations to Fredholm equations which we can solve. Existence, uniqueness, and analytical properties of the solutions are studied. An example is given for illustrating the results.

In a forthcoming paper, we use our method for solving the inverse-scattering problem at fixed energy in the case of a central plus a spin–orbit potential.

2. INTERPOLATION FORMULAS

A. Survey of Previous Results

Our aim is the study of regular solutions of the Schrödinger equation for potentials which depend linearly on λ. In previous papers[11–13] on λ-independent potentials, we used a method whose fundamental tool is the equation

$$K(r, r') = f(r, r') - \int_0^r K(r, \rho) f(\rho, r') \rho^{-2}\, d\rho, \quad (2.1)$$

previously used by Regge[2] and Newton.[3]

To a function $f_V^W(r, r')$, which is a solution of the partial differential equation

$$[D_V(r) - D_V(r')] f_V^W(r, r') = 0 \quad (2.2a)$$

and the boundary conditions

$$f_V^W(r, 0) = f_V^W(0, r') = 0, \quad (2.2b)$$

where[14]

$$D_V(r) \equiv r^2\left(\frac{\partial^2}{\partial r^2} + 1\right) - r^2 V(r) + \tfrac{1}{4}, \quad (2.3)$$

(2.1) associates a function $K_V^W(r, r')$, which is a solution of the partial differential equation

$$[D_W(r) - D_V(r')] K_V^W(0, r') = 0 \quad (2.4a)$$

and the boundary conditions

$$K_V^W(r, 0) = K_V^W(0, r') = 0, \quad (2.4b)$$

where

$$W(r) = V(r) - 2r^{-1}(d/dr) r^{-1} K_V^W(r, r). \quad (2.5)$$

Let us now denote by $\psi_\lambda(z)$ the "regular"[15] solutions of the Schrödinger equation for the potential $V(z)$, with the following normalization:

$$D_V(z)\psi_\lambda(z) = \lambda^2 \psi_\lambda(z),$$
$$(\tfrac{1}{2}\pi z)^{-\frac{1}{2}}\Gamma(1+\lambda)\left(\frac{z}{2}\right)^{-\lambda}\psi_\lambda(z) \to 1 \quad \text{as} \quad z \to 0. \quad (2.6)$$

The regular solutions $\chi_\lambda(z)$ corresponding to $W(z)$ can be generated in a simple way from $K_V^W(z, z')$:

$$\chi_\lambda(z) = \psi_\lambda(z) - \int_0^z K_V^W(z, \zeta)\psi_\lambda(\zeta)\zeta^{-2}\, d\zeta \quad (\operatorname{Re}\lambda \geq 0). \quad (2.7)$$

Equations (2.1)–(2.7) enable us therefore to solve completely the Schrödinger equation for a potential W defined from $f_V^W(r, r')$ through (2.1) and (2.5). For studying special classes of potentials, we used special choices of the input function $f_V^W(r, r')$. Such a choice involves first a definition of the starting potential $V(z)$. We call such a potential a "reference" or a "base" potential, and the corresponding regular solutions the "reference" solutions or "the base." The base which corresponds to $V(r) = 0$, viz., to the standard "fixed" kinetic energy, will be called "the standard base." The one corresponding to $V(r) = 1$, that is to say, to a Schrödinger equation without any energy term, will be called "the zero base." Suppose now we know, for $V(z)$, the function $\psi_\lambda(z)$ for any λ belonging to a domain E included in the half plane $\operatorname{Re}\lambda \geq \epsilon > 0$. Then a convenient choice for $f_V^W(r, r')$ is the following expansion, provided that it is convergent in the domain we need:

$$f_V^W(r, r') = \int_E \psi_\mu(r)\psi_\mu(r')\, d[\alpha(\mu)], \quad (2.8)$$

where $\alpha(\mu)$ is any piecewise-differentiable function of μ in E, or, more precisely,

$$f_V^W(r, r') = \int_{\mathbf{C}} \beta_\mu(W, V)\psi_\mu(r)\psi_\mu(r')\, d\mu + \sum_{\mathbf{S}} \gamma_{\mu_i}(W, V)\psi_{\mu_i}(r)\psi_{\mu_i}(r'), \quad (2.9)$$

where **C** is a set of arcs of differentiable curves and **S** a set of isolated points μ_i in E. So as to obtain a real potential, assuming $V(r)$ is itself real, **C** and **S** should

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[11] P. C. Sabatier, J. Math. Phys. **8**, 905 (1967).

[12] P. C. Sabatier, "Interpolation Formulas in the Angular Momentum Plane," Preprint, Orsay TH/180 (1967); J. Math. Phys. (to be published).

[13] P. C. Sabatier, Preprint Orsay TH/200 5p.; Compt. Rend. (to be published).

[14] The actual potential energy is obtained from $V(r)$ by multiplication by the standard energy $(2m)^{-1}h^2k_0^2$. Referring ourselves to Footnote 4, we see that a further implicit dependence on E comes in because r is measured in units of k_0^{-1}. With these notations, the Schrödinger equation (2.6) at a kinetic energy $h^2k^2/2m$ is obtained by setting $V(r) = 1 - k^2/k_0^2$.

[15] In the following, we use the word "regular" for the solutions associated with a potential like $\psi_\lambda(z)$ is associated with $V(z)$. It is needless to say that these solutions are regular at the origin only for $\operatorname{Re}\lambda > -\frac{1}{2}$.

be symmetric with respect to the real axis, whereas β_μ and γ_μ should take conjugate values at conjugate complex points. Further restrictions on **S** and **C** enabled us to study more thoroughly special classes of potentials. For instance, if **C** is empty and if **S** is the set Q of positive rational numbers, we obtain the class A of potentials which are analytic functions of a rational power of z (say ρ) in a nonvanishing circle centered at the origin in the ρ complex plane—at least provided we can find ϵ such that

$$|\gamma_\mu(W, V)| < |\Gamma(1+\mu)|^2\epsilon^{-2\mu}; \quad \forall\mu \in Q. \qquad (2.10)$$

The restriction of **S** to the set of positive integers and half-integers leads us to analytic potentials. For potentials of class A, we showed the linear formula:

$$\gamma_\mu(W, V_0) - \gamma_\mu(V, V_0) = \gamma_\mu(W, V). \qquad (2.11)$$

This property applies[16] in fact to classes much larger than A and to coefficients β_μ. The formula (2.11) can be used to express $\gamma_\mu(W, V)$ as a difference between $\gamma_\mu(W, 1)$ and $\gamma_\mu(V, 1)$. These coefficients are associated to potentials V and W, respectively, with the zero base. It turns out that for integer μ's they are equal to the proportionality coefficients $\gamma_n(\psi)$ and $\gamma_n(\chi)$ which we defined in the Introduction. It is also possible to associate all the coefficients of rational indices with analogous properties. We gave several properties and examples of the γ_μ's and showed that $K_V^W(z, z')$ and also the resolvent of (2.1) can be expanded like $f_V^W(z, z')$—for instance, for a potential of class A, we get[12]

$$K_V^W(z, z') = \sum_Q \gamma_{\mu_i}(W, V)\chi_{\mu_i}(z)\psi_{\mu_i}(z'). \qquad (2.12)$$

Insertion of (2.12) in (2.7) yields for $\chi_\lambda(z)$ an expansion formula along the $\chi_\mu(z)$, whose coefficients are Wronskians of $\psi_\lambda(z)$ and $\psi_\mu(z)$. This formula is nothing but (1.3).

The method for dealing with the Schrödinger equation for λ-dependent potentials will run as follows. First, we use an elementary method in order to obtain the simplest interpolation formula. This formula yields the special analytic properties of the regular wavefunctions in the λ plane, which can be used in turn so as to obtain more sophisticated interpolation formulas, from which we deduce the integral equations. The form of these equations allows us to deal with more general classes of potentials than those used in the first steps.

Notice: We give a figure (Fig. 1) for illustrating the machineries used in Sec. 2A and in the following. As regards Sec. 2A, the input function is $f(r, r')$. The properties (1) are (2.2) and (2.3). The output generator is $K(r, r')$, the property (2) is (2.4), Relation A is (2.1). B is the formula obtained through replacing $\psi_\lambda(z)$ by $\chi_\lambda(z)$ in (2.6). C is (2.5), D is (2.7). E is the irreversible step of taking the asymptotic behavior of the wavefunctions. F has been studied in Ref. 3.

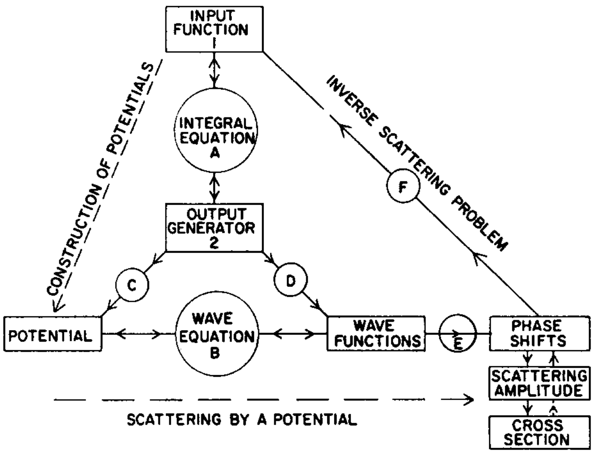

FIG. 1. This scheme shows the logic of the machinery recalled in Sec. 2. It applies also to the Gel'fand–Levitan method, as well as to the method given in the present paper for scattering by a spin-orbit potential. The circles are used for relations between quantities and are denoted by letters. The numbers 1 and 2 refer to specific properties of the quantities of interest. The correspondence between these symbols and formulas of the paper is given at the end of Secs. 2A and 3F. The dotted line in the step "cross section → scattering amplitude" means that this is in no way a closed problem.

B. Expansions in Terms of Wavefunctions

So as to be able to devise an elementary method for getting at interpolation formulas, we have first to study the analytic properties of wavefunctions corresponding to a potential of class A for all the μ_i's involved in the set S of indices which characterizes the subclass of A we are dealing with. We first limit our study to analytic potentials for which all the mathematical difficulties involved in the problem are exhibited. Simple changes of variables and functions, similar to those stated in a previous paper,[17] enable us then to relate the problem involving any potential of class A to a problem involving an analytic potential.

Let us denote by $d_\mu(z)$, $(\mu \geq 0)$, the quotient of $\chi_\mu(z)$ by $(\frac{1}{2}\pi z)^{\frac{1}{2}}(z/2)^\mu/\Gamma(1+\mu)$. It follows from (2.6) that $d_\mu(z)$ obeys the following equation:

$$Ld_\mu(z) \equiv \left[z\frac{d}{dz}z\frac{d}{dz} + z^2\mathbf{W}(z) + 2\mu z\frac{d}{dz}\right]d_\mu(z) = 0, \qquad (2.13a)$$

where

$$\mathbf{W}(z) = 1 - W(z) \qquad (2.13b)$$

[16] See the remark on the end of Sec. 3D. The same property holds for the spectral function in the Gel'fand–Levitan procedure.

[17] Reference 11, Appendix A. Notice that the index μ which is used hereafter is not equal to the index λ used in Ref. 11, but to $\lambda + \frac{1}{2}$.

can depend on μ and $z\mathbf{W}(z)$ is analytic in a circle $|z| < R_0 > 0$. It is easy to apply Frobenius's method[18] to this equation, giving $d_\mu(z)$ the form

$$d_\mu(z) = \sum_p d_\mu^p z^p \quad (d_\mu^0 = 1) \tag{2.14}$$

and setting

$$z^{-p}Lz^p = \sum_{n=0}^{\infty} g_\mu^n(p) z^n = p(2\mu + p) + z^2\mathbf{W}(z).$$

Equation (2.13a) is equivalent to the system

$$\begin{aligned} &d_\mu^0 = 1, \\ &d_\mu^1 g_\mu^0(1) + d_\mu^0 g_\mu^1 = 0, \\ &\cdot \\ &\cdot \\ &\cdot \\ &d_\mu^p g_\mu^0(p) + d_\mu^{p-1} g_\mu^1 + \cdots d_\mu^0 g_\mu^p = 0, \end{aligned}$$

where we used a condensed notation for the $g_\mu^n(p)$ with $n \geq 1$, since they do not depend on p. For potentials that depend linearly on μ, we can find an upper bound of the form $(2\mu + 1)M(R)$ for $|W'(z)|$ on the circle $|z| = R < R_0$. The Cauchy theorem gives then upper bounds for $g_\mu^n(p)$:

$$g_\mu^0(p) = p(2\mu + p), \quad g_\mu^{n+1} \leq [M(R)/R^n](2\mu + 1).$$

Upper bounds D_μ^{p+1} for $|d_\mu^{p+1}|$ are therefore defined by the system

$$\begin{aligned} D_\mu^{p+1} &= \frac{M(R)(2\mu + 1)}{(p + 1)(2\mu + p + 1)} \\ &\quad \times \{D_\mu^p + D_\mu^{p-1}R^{-1} + \cdots D_\mu^0 R^{-p}\}, \\ &\cdot \\ &\cdot \\ &\cdot \\ D_\mu^1 &= M(R) D_\mu^0 = M(R), \end{aligned}$$

whose solution is

$$D_\mu^{p+1} = \prod_{n=0}^{n=p} R^{-1}\left[\frac{n(R)(2\mu + 1)}{(n + 1)(2\mu + n + 1)} + \frac{n(2\mu + n)}{(n + 1)(2\mu + n + 1)}\right], \tag{2.15}$$

where $n(R)$ is equal to $RM(R)$. A more convenient upper bound can be obtained by replacing n by $n(R)$ in the numerator of all terms whose index n is smaller than $n(R)$, and replacing $n(R)$ by n in the numerator of terms whose index n is larger or equal to $n(R)$, whereas $(4\mu + n + 1)/(2\mu + n + 1)$ is replaced everywhere by 2. Use of Stirling's formula then leads us to the bound

$$D_\mu^{p+1} < (2/R)^p \exp [n(R)]. \tag{2.16a}$$

[18] E. L. Ince, *Ordinary Differential Equations* (Dover Publications, Inc., New York, 1956), p. 396.

It is needless to say that it is possible to manage (2.15) so as to get bounds for $|d_\mu^{p+1}|$, which yield for (2.14) a radius of convergence equal to R, and not $R/2$, as one might deduce from (2.16); but those bounds are *not* independent of μ. Let us now denote by $\bar{d}_{2\mu}(z)$ the product of two functions $d_\mu^{(1)}(z)$ and $d_\mu^{(2)}(z)$, which correspond to two different potentials, analytic in circles $|z| < R_1$ and $|z| < R_2$, respectively, with $R_1 \leq R_2$. It is a matter of simple algebraic evaluations to show that the following bound applies to $\bar{d}_{2\mu}^{p+1}$:

$$|\bar{d}_{2\mu}^{p+1}| < (2/R)^p C(R), \tag{2.16b}$$

provided that $R = R_1 - \epsilon$ and $C(R)$ is conveniently chosen. Let us now introduce a function $f(z)$, analytic in a circle $|z| < \gamma$, which can be chosen smaller than $R/2$. It is possible to find a positive number $P(\gamma)$ such that

$$|\bar{d}_n^{p+1}| < \gamma^{-p}P(\gamma) \quad |f_k| < \gamma^{-k}P(\gamma), \tag{2.17}$$

where f_k is the kth coefficient of the Taylor expansion of $f(z)$.

Let us now try to expand $f(z)$ in terms of the functions $z^n\bar{d}_n(z)$:

$$f(z) \overset{(?)}{=} \sum b_n z^n \bar{d}_n(z). \tag{2.18}$$

Formal identification of coefficients leads us to the equations

$$b_k = f_k - \sum_{n=0}^{k-1} b_n \bar{d}_n^{k-n}, \tag{2.19}$$

which enable one to get any coefficient b_n by induction. Upper bounds for the $|b_n|$ can obviously be obtained by replacing any term in (2.19) by the upper bound of its absolute value. This yields the following bounds:

$$|b_k| < \gamma^{-k}[1 + P(\gamma)]^{k-1}f_0$$

or

$$|b_k| < C[Q(\gamma)]^{-k}.$$

It should be noticed that $Q(\gamma)$ is smaller than γ. Now it follows from (2.17) that

$$|z^n\bar{d}_n(z)| < P(\gamma)\,|z|^n\,[1 - |z|\,\gamma^{-1}]^{-1}.$$

As a result, provided z is in the circle $|z| < Q(\gamma) - \epsilon$, the series $\sum_0^\infty b_n z^n \bar{d}_n(z)$ is absolutely bounded, uniformly in z, by the convergent series

$$P(\gamma)[1 - |z|\,\gamma^{-1}]^{-1} \sum_0^\infty |Q(\gamma)|^{-n}\,|z|^n.$$

Since every term in the series (2.18) is an analytic function of z, the sum of the series is analytic in z and defines an analytic function equal to $f(z)$. We have therefore proved the following theorem, valid for potentials W_1 and W_2 which depend linearly on λ.

Expansion theorem: Given $z\mathbf{W}^{(1)}(z)$ and $z\mathbf{W}^{(2)}(z)$ analytic in the circles $|z| < R_1$ and $|z| < R_2$, respectively, and given a function $f(z)$ analytic in the circle $|z| < R$, we can find a circle Ω, centered at the origin, with nonvanishing radius ω (smaller than R_1, R_2, R), in which $f(z)$ can be expanded uniformly in z in terms of the products of the functions $z^{n/2}d^{(1)}_{n/2}(z)$ and $z^{n/2}\, d^{(2)}_{n/2}(z)$, which correspond to $\mathbf{W}^{(1)}$ and $\mathbf{W}^{(2)}$:

$$f(z) = \sum_0^\infty a_{n/2}z^{n/2}d^{(1)}_{n/2}(z)z^{n/2}d^{(2)}_{n/2}(z) \quad (z \in \Omega). \tag{2.20a}$$

Use of the methods quoted above enables one to prove that if $z^{(2-m^{-1})}\mathbf{W}^{(1)}(z)$ and $z^{(2-m^{-1})}\mathbf{W}^{(2)}(z)$ are analytic functions of the principal[19] determination ζ of $z^{(m^{-1})}$ in circles $|\zeta| < R_1$ and $|\zeta| < R_2$, and being given a function $f(z)$ analytic in ζ for $|\zeta| < R$, we can find a circle Ω in the ζ plane, centered at the origin, in which

$$f(z) = \sum_{S^*} a_\mu z^\mu d^{(1)}_\mu(z)z^\mu d^{(2)}_\mu(z), \tag{2.20b}$$

where S^* includes all numbers of the form $k/2m$, k being a positive integer or 0, m being a positive integer.[20,21]

C. Simplest Interpolation Formula

Let us introduce the following notation:

$$s_\lambda(z) = (\tfrac{1}{2}\pi z)^{\frac{1}{2}}[\Gamma(1+\lambda)]^{-1}(z/2)^\lambda, \tag{2.21}$$

$$\tau_\lambda(z) = 2\pi^{-1}z^{-1}\chi_\lambda(z)s_\lambda(z), \tag{2.22}$$

$$\sigma_\lambda(z) = 2\pi^{-1}z^{-1}\chi_\lambda(z)s_{-\lambda}(z), \tag{2.23}$$

$$D_z = \frac{d^2}{dz^2} + z^{-1}\frac{d}{dz}. \tag{2.24}$$

$s_\lambda(z)$ is the regular wavefunction which corresponds to a potential equal to 1 and, therefore, to the zero base defined in Sec. 2A. It follows from (2.6) and (2.26) that $\tau_\lambda(z)$ and $\sigma_\lambda(z)$ are solutions of the equations

$$\left[D_z + \mathbf{W}(z) - 2\lambda z^{-1}\frac{d}{dz}\right]\tau_\lambda(z) = 0, \tag{2.25}$$

$$\left[D_z + \mathbf{W}(z) + 2\lambda z^{-1}\frac{d}{dz}\right]\sigma_\lambda(z) = 0, \tag{2.26}$$

in which $\mathbf{W}(z)$ is defined as in (2.13b). The equation (2.26) is identical with (2.13a), and $\sigma_\lambda(z)$ is equal to $(\pi\lambda)^{-1}\sin\pi\lambda\, d_\lambda(z)$.

Let us now assume that $\mathbf{W}(z)$ can take the two following values:

$$\mathbf{W}^\pm(z) = \mathbf{U}(z) \mp 2\lambda Q(z), \tag{2.27}$$

where neither $\mathbf{U}(z)$ nor $Q(z)$ depend on λ, and let us introduce the operator

$$T_z = D_z + \mathbf{U}(z). \tag{2.28}$$

We define the two set of solutions (+) and (−) of (2.15) and (2.16):

$$T_z\tau^+_\lambda(z) = 2\lambda\left[z^{-1}\frac{d}{dz} + Q(z)\right]\tau^+_\lambda(z), \tag{2.29}$$

$$T_z\sigma^+_\lambda(z) = 2\lambda\left[-z^{-1}\frac{d}{dz} + Q(z)\right]\sigma^+_\lambda(z), \tag{2.30}$$

$$T_z\tau^-_\lambda(z) = 2\lambda\left[z^{-1}\frac{d}{dz} - Q(z)\right]\tau^-_\lambda(z), \tag{2.31}$$

$$T_z\sigma^-_\lambda(z) = 2\lambda\left[-z^{-1}\frac{d}{dz} - Q(z)\right]\sigma^-_\lambda(z). \tag{2.32}$$

As we shall see below (Sec. 2F), this separation has a direct physical meaning in scattering problems of a spinning particle. Assume now for convenience that $z\mathbf{W}^\pm(z)$ is analytic for $|z| < R$, and S is the set of integers and half-integers. We assert that the following expansions hold in a nonvanishing circle:

$$\Omega(z \in \Omega \mid |z| < \omega):$$

$$\frac{\pi\lambda}{\sin\pi\lambda}\,\sigma^+_\lambda(z) = \sum_{\mu\in S^*}\frac{\lambda}{\lambda+\mu}\,a^+_\mu\tau^-_\mu(z), \tag{2.33}$$

$$\frac{\pi\lambda}{\sin\pi\lambda}\,\sigma^-_\lambda(z) = \sum_{\mu\in S^*}\frac{\lambda}{\lambda+\mu}\,a^-_\mu\tau^+_\mu(z), \tag{2.34}$$

where a^+_μ and a^-_μ are the expansion coefficients in terms of $\tau^-_\mu(z)$ and $\tau^+_\mu(z)$ of the functions $F^+(z)$ and $F^-(z)$, respectively, defined by

$$F^\pm(z) = \exp\left[\pm\int_0^z \tau Q(\tau)\,d\tau\right] = \sum_{\mu\in S} a^\pm_\mu\tau^\mp_\mu(z), \tag{2.35}$$

where the indices $\pm$ correspond to each other according to their position. This convention will be used in the following.

Proof: The functions $\tau^\pm_\mu(z)$ have the same form as the functions $z^{2\mu}d_{2\mu}(z)$ introduced in Sec. 2B. It is therefore possible to define a set of coefficients a^+_μ and a^-_μ, and to find a circle $\Omega(z \in \Omega|\, |z| < \omega < R)$ in which the series (2.33)–(2.35) are analytic, whereas the functions on the left-hand side of these formulas are also analytic. So as to prove the equality (2.33), we need only to apply $T_z - 2\lambda[Q(z) - z^{-1}(d/dz)]$ to the right-hand side of (2.33) and use (2.31). We obtain by this way $2(z^{-1}(d/dz) - Q(z))F^+(z)$, which is equal to zero, according to (2.35). Comparison of the boundary conditions at the origin completes the proof. The proof of (2.34) is similar.

[19] We thus identify the determination which is real and positive when z is real and positive.

[20] Some symbols and notations which have been introduced for convenience in the unnumbered formulas of this section will be used with a different meaning in the following.

[21] The set S^* contains the set S and the element zero.

Coming back to the notations used in Sec. 2A, we write (2.33) and (2.34) as

$$\chi_\lambda^\pm(z)s_{-\lambda}(z) = \frac{\sin \pi\lambda}{\pi\lambda} \sum_{\mu\in S} \frac{\lambda}{\lambda+\mu} a_\mu^\pm \chi_\mu^\mp(z) s_\mu(z). \quad (2.36)$$

Formulas (2.36) can be considered as the simplest interpolation formulas for the regular solutions of the Schrödinger equation whose potential depends linearly on λ. The analogous formula for the λ-independent potential has been derived in a previous paper,[12] and is easily obtained by putting (+) equal to (−) in (2.36), or $Q(r)$ equal to zero in (2.35). In order to make the analogy more obvious, let us introduce the notation

$$\gamma_\mu(W^\pm, 1) = 2\mu\pi^{-1} a_\mu^\pm. \quad (2.37)$$

(2.36) therefore takes the form

$$\frac{\pi\lambda}{\sin \pi\lambda} \chi_\lambda^\pm(z) s_{-\lambda}(z)$$
$$= \chi_0(z)s_0(z) + \sum_{\mu\in S} \frac{\pi}{2\mu} \chi_\mu^\mp(z) s_\mu(z) \frac{\lambda}{\lambda+\mu} \gamma_\mu(W^\pm, 1). \quad (2.38)$$

(2.38) is obviously an interpolation formula on the zero base. Formulas (2.36)–(2.38) are straightforwardly generalized, if $z^{2-m^{-1}}\mathbf{W}^\pm(z)$ is an analytic potential of ζ $(= z^{m^{-1}})$ by, allowing S to include all numbers of the form $k/2m$.

D. Analytical Properties of Wavefunctions in the λ Plane

Even Potentials

If both $\mathbf{U}(z)$ and $Q(z)$ are even functions, integer indices only are involved in the right-hand side of (2.36). This formula defines therefore $\chi_\lambda^\pm(z)s_{-\lambda}(z)$ as entire functions of λ, which are uniformly bounded in the strip $|\mathrm{Im}\,\lambda| < \epsilon$, and obviously go[22] to $C(z)\,|\sin \pi\lambda/\pi\lambda|$ as $|\lambda|$ goes to infinity outside of this strip. The functions $\lambda\chi_\lambda^\pm(z)s_{-\lambda}(z)$ are therefore entire functions of λ of order 1, type π, bounded on the real axis, and going to zero like $C\lambda$ as λ goes to zero. The asymptotic behavior of $\chi_\lambda^\pm(z)$ as $|\lambda| \to \infty$ outside the negative real axis is

$$\chi_\lambda^\pm(z) = (\tfrac{1}{2}\pi z)^{\frac{1}{2}} [\Gamma(1+\lambda)]^{-1} (z/2)^\lambda$$
$$\times [F^\pm(z) - \lambda^{-1} z^{-1} G^\pm(z) + O(\lambda^{-2})], \quad (2.39)$$

where

$$G^\pm(z) = 2\pi^{-1} \sum_{n=1}^\infty n a_n^\pm \chi_n^\mp(z) s_n(z). \quad (2.40)$$

The values of $\chi_\lambda^\pm(z)$ for integer λ are related to each other in the following way:

$$\chi_{-p}^\pm(z) = (-1)^p a_p^\pm \chi_p^\mp(z), \quad (2.41)$$

where

$$p = 1, 2, 3, \cdots. \quad (2.42)$$

Analytic and More General Potentials

For potentials which are analytic functions of $z^{m^{-1}}$, let us introduce the function

$$\chi_{\lambda,m}^\pm(z) = \frac{\sin 2m\pi\lambda}{2m \sin \pi\lambda} \chi_\lambda^\pm(z). \quad (2.43)$$

It is easy to see that the functions $\lambda\chi_{\lambda,m}^\pm(z)s_{-\lambda}(z)$ are entire functions of λ of order 1, type $2m\pi$, bounded on the real axis, and going to zero like $C\lambda$ as λ goes to zero. The asymptotic behavior (2.39) still holds, with

$$G^\pm(z) = \sum_{\mu\in S} \gamma_\mu(W^\pm, 1) \chi_\mu^\mp(z) s_\mu(z), \quad (2.44)$$

and the values of the functions $\chi_{\lambda,m}^\pm(z)$ at points of S and opposite points are related to the values of $\chi_\lambda^\pm(z)$ by the relations

$$\chi_{k/2m,m}^\pm(z) = 0, \quad \text{for integer } k \neq 2m, \quad (2.45)$$

$$\chi_{-k/2m,m}^\pm(z) = (-1)^k a_{k/2m}^\pm \chi_{k/2m}^\mp(z), \quad (2.46)$$

$$\chi_{-p,m}^\pm(z) = a_p^\pm \chi_p^\mp(z) = (-1)^p a_p^\pm \chi_{p,m}^\mp(z). \quad (2.47)$$

Evaluation of $G^\pm(z)$

Application of zT_z, defined by 2.28, to $F^\pm(z)$ as given by (2.35) leads us to the formula

$$\tfrac{1}{2} z T_z F^\pm(z) = [d/dz \mp zQ(z)](z^{-1}G^\pm(z)). \quad (2.48)$$

Derivation of $G^\pm(z)$ from (2.48) is straightforward and leads us to the formula

$$G^\pm(z) = \tfrac{1}{2} z F^\pm(z) \Big[\pm z^2 Q(z) + \int_0^z t^3 Q^2(t)\, dt + \int_0^z t\mathbf{U}(t)\, dt \Big]. \quad (2.49)$$

These formulas are generalizations of the formulas giving $K_1^W(z,z)$ in our previous paper.[12] Data of G^+, G^-, F^+, and F^- enable one easily to extract $Q(z)$ (as $z^{-3}[G^+/F^+ - G^-/F^-]$) and therefore $U(z)$. It is more difficult, but still possible, to extract Q and U from the data of G^+ and G^- only.

E. Further Interpolation Formulas

We deal with potentials whose product by z is analytic, since they enable one to encounter all difficulties involved in potentials of class A. We introduce a λ-independent potential $V(z)$ whose regular wavefunctions are $\psi_\lambda(z)$. We use for $\chi_{\lambda,1}^\pm(z)$ and $\psi_{\lambda,1}^\pm(z)$ the more convenient notation $\bar{\chi}_\lambda^\pm(z)$ and $\bar{\psi}_\lambda(z)$ [therefore equal to $\cos \pi\lambda\psi_\lambda(z)$].

[22] In this section, $C(z)$ is any finite function of z. It is not meant to have the same value every time that it is used.

So as to obtain interpolation formulas, we apply the Lagrange–Valiron theorem[23] to a function $\lambda g(\lambda)$ which is entire, of order 1, finite type 2π, bounded on the real axis, and going to zero like $C\lambda$ as λ goes to zero. As a result, $g(\lambda)$ has the representation

$$g(\lambda) = \cos \pi\lambda \frac{\sin \pi\lambda}{\pi\lambda} \times \left\{ g(0) + \sum_1^\infty \left[\frac{\lambda}{\lambda - p} g(p) + \frac{\lambda}{\lambda + p} g(-p) \right] - \sum_0^\infty \left[\frac{\lambda}{\lambda - (p + \frac{1}{2})} g(p + \tfrac{1}{2}) + \frac{\lambda}{\lambda + p + \frac{1}{2}} g(-p - \tfrac{1}{2}) \right] \right\}. \tag{2.50}$$

Cross-Products Interpolation Formulas

Let us first recall[12] the reflection formula for $\psi_\lambda(z)$, which is a special case of (2.46) and (2.47),

$$\bar{\psi}_{-\frac{1}{2}k}(z) = (-1)^k b_{\frac{1}{2}k} \psi_{\frac{1}{2}k}(z), \tag{2.51}$$

where

$$b_\mu = \tfrac{1}{2}\pi\mu^{-1}\gamma_\mu(V, 1). \tag{2.52}$$

The asymptotic behavior of $\psi_\lambda(z)$ in the λ plane, outside the negative real axis, is given by the following formula, which is a special case of (2.39),

$$\psi_\lambda(z) = (\tfrac{1}{2}\pi z)^{\frac{1}{2}} [\Gamma(1 + \lambda)]^{-1} (z/2)^\lambda \times [1 - \lambda^{-1} z^{-1} K_1^V(z, z) + O(\lambda^{-2})]. \tag{2.53}$$

$\chi_\lambda^\pm(z)$ has poles for negative half-integers, whereas $\psi_{-\lambda}(z)$ has poles for positive half-integers. This remark and consideration of the asymptotic behavior of (2.39) and (2.53) show that $\cos \pi\lambda \chi_\lambda^\pm(z)\psi_{-\lambda}(z)$ has all the properties required for the function $g(\lambda)$ as defined above, with

$$g^\pm(\tfrac{1}{2}k) = (-1)^k \chi_{\frac{1}{2}k}^\pm(z) \psi_{\frac{1}{2}k}(z) b_{\frac{1}{2}k}, \tag{2.54}$$

$$g^\pm(-\tfrac{1}{2}k) = (-1)^k \chi_{\frac{1}{2}k}^\mp(z) \psi_{\frac{1}{2}k}(z) a_{\frac{1}{2}k}^\pm. \tag{2.55}$$

Insertion of (2.54) and (2.55) in (2.50) and simplification by $\cos \pi\lambda$ lead us to the cross-product "unsubtracted" formula:

$$\chi_\lambda^\pm(z)\psi_{-\lambda}(z) = \frac{\sin \pi\lambda}{\pi\lambda} \left\{ \sum_{\mu \in S^*} \chi_\mu^\mp(z)\psi_\mu(z) \frac{\lambda}{\lambda + \mu} a_\mu^\pm + \sum_{\mu \in S} \chi_\mu^\pm(z)\psi_\mu(z) \frac{\lambda}{\lambda - \mu} b_\mu \right\}. \tag{2.56}$$

Comparison[24] of the asymptotic behaviors of both sides of (2.56) yields the formula

$$(\tfrac{1}{2}\pi z) F^\pm(z) = \sum_{\mu \in S^*} a_\mu^\pm \chi_\mu^\mp(z)\psi_\mu(z) + \sum_{\mu \in S} b_\mu \chi_\mu^\pm(z)\psi_\mu(z). \tag{2.57}$$

Subtraction of (2.57) from (2.56) yields the subtracted cross product formula. Comparison[24] of the coefficients of λ^{-1} in (2.56) yields the formula

$$\sum_{\mu \in S} [\gamma_\mu(V, 1)\chi_\mu^\pm(z) - \gamma_\mu(W^\pm, 1)\chi_\mu^\mp(z)]\psi_\mu(z) = -\tfrac{1}{2} z F^\pm(z) \left[\pm z^2 Q(z) + \int_0^z t^3 Q^2(t)\, dt + \int_0^z t(\mathrm{U} - \mathrm{V})\, dt \right]. \tag{2.58}$$

If we refer ourselves to the argument given in our previous paper for obtaining the linear formula (2.11), we see that it fails in the present case.

Formulas (2.56) and (2.57) can be straightforwardly generalized, using the appropriate sequence S, for all potentials of A. They are valid in a domain of the z plane including Ω.

Wronskian Interpolation Formulas

Let us now introduce the function

$$\rho_{-\lambda}(z) = \frac{\psi_{-\lambda}(z)\psi_\mu'(z) - \psi_{-\lambda}'(z)\psi_\mu(z)}{\mu + \lambda}, \tag{2.59}$$

where μ is a given number $\notin S$. For analytic potentials, the function $\rho_{-\lambda}(z)$ has poles only for positive half-integers, and its asymptotic behavior is

$$\rho_{-\lambda}(z) \sim (\tfrac{1}{2}\pi z)^{\frac{1}{2}} (z/2)^{-\lambda} [\Gamma(1 - \lambda)]^{-1} z^{-1} \psi_\mu(z) \times [1 + O(\lambda^{-1})]. \tag{2.60}$$

The function $\cos \pi\lambda \chi_\lambda^\pm(z)\rho_{-\lambda}(z)$ has therefore all the properties required for the function $g(\lambda)$ as defined above, with

$$g^\pm(\tfrac{1}{2}k) = (-1)^k \chi_{\frac{1}{2}k}^\pm \frac{\psi_{\frac{1}{2}k}\psi_\mu' - \psi_{\frac{1}{2}k}'\psi_\mu}{\mu + \frac{1}{2}k} b_{\frac{1}{2}k}, \tag{2.61a}$$

$$g^\pm(-\tfrac{1}{2}k) = (-1)^k \chi_{\frac{1}{2}k}^\mp \frac{\psi_{\frac{1}{2}k}\psi_\mu' - \psi_{\frac{1}{2}k}'\psi_\mu}{\mu - \frac{1}{2}k} a_{\frac{1}{2}k}^\pm. \tag{2.61b}$$

Insertion of (2.61) in (2.50) leads us to

$$\frac{\pi\lambda}{\sin \pi\lambda} \chi_\lambda^\pm \frac{\psi_{-\lambda}\psi_\mu' - \psi_{-\lambda}'\psi_\mu}{\mu + \lambda} = \chi_0 \frac{\psi_0\psi_\mu' - \psi_0'\psi_\mu}{\mu} + \sum_{\nu \in S} \frac{\lambda}{\lambda - \nu} b_\nu \chi_\nu^\pm \frac{\psi_\nu\psi_\mu' - \psi_\nu'\psi_\mu}{\mu + \nu} + \sum_{\nu \in S} \frac{\lambda}{\lambda + \nu} a_\nu^\pm \chi_\nu^\mp \frac{\psi_\nu\psi_\mu' - \psi_\nu'\psi_\mu}{\mu - \nu}. \tag{2.62}$$

Let us now equate[24] the limits obtained in the two

[23] R. P. Boas, Jr., *Entire Functions* (Academic Press Inc., New York, 1954).
[24] For the legitimacy of these comparisons, see the Appendix to Ref. 12.

sides of (2.62) by letting $|\lambda| \to \infty$, with $|\mathrm{Im}\,\lambda| > \epsilon$,

$$\frac{\pi}{2} F^{\pm}\psi_{\mu} = \chi_0 \frac{\psi_0\psi'_{\mu} - \psi'_0\psi_{\mu}}{\mu} + \sum_{\nu \in S} \left[\frac{b_{\nu}}{\mu+\nu}\chi_{\nu}^{\pm} + \frac{a_{\nu}^{\pm}}{\mu+\nu}\chi_{\nu}^{\mp}\right](\psi_{\nu}\psi'_{\mu} - \psi'_{\nu}\psi_{\mu}). \tag{2.63}$$

Subtraction of (2.63) from (2.62) yields

$$\frac{\pi\lambda}{\sin \pi\lambda}\chi_{\lambda}^{\pm}\frac{\psi_{-\lambda}\psi'_{\mu} - \psi'_{-\lambda}\psi_{\mu}}{\mu+\lambda} - \frac{\pi}{2}F^{\pm}\psi_{\mu} = \sum_{\nu \in S}\left[\frac{\nu b_{\nu}\chi_{\nu}}{(\lambda-\nu)(\mu+\nu)} - \frac{\nu a_{\nu}^{\pm}\chi_{\nu}^{\mp}}{(\lambda+\nu)(\mu-\nu)}\right] \times (\psi_{\nu}\psi'_{\mu} - \psi'_{\nu}\psi_{\mu}). \tag{2.64}$$

The two sides of (2.64) are entire functions of μ. We can therefore make $\mu = \lambda$. Using the Wronskian properties for the coefficient of $\chi_{\lambda}^{\pm}$ and coming back to the notations of (2.37), we get

$$\chi_{\lambda}^{\pm} = F^{\pm}\psi_{\lambda} + \sum_{\nu \in S}\frac{\psi_{\nu}\psi'_{\lambda} - \psi'_{\nu}\psi_{\lambda}}{\lambda^2 - \nu^2} \times [\gamma_{\nu}(V, 1)\chi_{\nu}^{\pm} - \gamma_{\nu}(W^{\pm}, 1)\chi_{\nu}^{\mp}]. \tag{2.65}$$

In the same way, we can derive easily the unsubtracted interpolation formula

$$\frac{\pi}{2}\chi_{\lambda}^{\pm}(z) = \chi_0(z)\frac{\psi_0(z)\psi'_{\lambda}(z) - \psi'_0(z)\psi_{\lambda}(z)}{\lambda} + \lambda\sum_{\mu \in S}\frac{\psi_{\mu}\psi'_{\lambda} - \psi'_{\mu}\psi_{\lambda}}{\lambda^2 - \mu^2}[b_{\mu}\chi_{\mu}^{\pm}(z) + a_{\mu}^{\pm}\chi_{\mu}^{\mp}(z)]. \tag{2.66}$$

Formulas (2.65) and (2.66) can be generalized for all potentials of class A by taking the appropriate sequence S. They are valid in a domain of the z plane which contains at least Ω.

F. Physical Meaning of the Splitting

Let us study the scattering of a spin-$\frac{1}{2}$ particle by a central and a spin–orbit field. Assume the interaction is written as

$$V = (h^2/2m)[U_c(r) + 2\mathbf{l}\cdot\mathbf{s}U_s(r)], \tag{2.67}$$

so that the Schrödinger equation reads

$$\Delta\psi(\mathbf{r}) + [k^2 - U_c(r) - 2\mathbf{l}\cdot\mathbf{s}U_s(r)]\psi(\mathbf{r}) = 0. \tag{2.68}$$

It is well-known[25] that the differential cross section is equal to

$$I(\theta) = |f|^2 + |g|^2, \tag{2.69}$$

where

$$f(\theta) = (2ik)^{-1}\sum_0^{\infty}\{(l+1)[\exp(2i\delta_l^+) - 1] + l[\exp(2i\delta_l^-) - 1]\}P_l(\cos\theta), \tag{2.70}$$

$$g(\theta) = (2ik)^{-1}\sum_0^{\infty}[\exp(2i\delta_l^+) - \exp(2i\delta_l^-)]P_l^1(\cos\theta), \tag{2.71}$$

and the phase-shifts δ_l^+ and δ_l^- are given, respectively, by the asymptotic behavior, for $r \to \infty$, of the regular solutions of the following equations, in which $\lambda = l + \frac{1}{2}$:

$$(d^2/dr^2)\psi_{\lambda}^+(r) + [k^2 - U_c + U_s - \lambda U_s - (\lambda^2 - \tfrac{1}{4})r^{-2}]\psi_{\lambda}^+(r) = 0, \tag{2.72}$$

$$(d^2/dr^2)\psi_{\lambda}^-(r) + [k^2 - U_c + U_s + \lambda U_s - (\lambda^2 - \tfrac{1}{4})r^{-2}]\psi_{\lambda}^-(r) = 0. \tag{2.73}$$

It is easy to write these equations as a Schrödinger equation in which the potential W can take the two values given in (2.27) by setting

$$\mathrm{U}(z) = k^2 - U_c + U_s, \tag{2.74}$$

$$Q(z) = \tfrac{1}{2}U_s. \tag{2.75}$$

Potentials which depend linearly on λ occur also in the scattering of a spin-$\frac{1}{2}$ particle by a scattering center with spin.

3. INTEGRAL EQUATIONS

In this section, we successively derive integral equations starting from a zero energy and a finite energy base. It is shown that solving these equations can be reduced to solving a set of Fredholm equations. The existence and uniqueness of solutions is proved when certain conditions are fulfilled. We then show how these equations can be used to generate potentials belonging to classes larger than A.

A. Zero Energy Base

From (2.35) and (2.36) we obtain straightforwardly the subtracted interpolation formula:

$$\chi_{\lambda}^{\pm}(z)\left[\frac{\pi\lambda}{\sin\pi\lambda}s_{-\lambda}(z)\right] = \frac{\pi}{2}zF^{\pm}(z) - \sum_{\mu \in S}\frac{\mu}{\lambda+\mu}a_{\mu}^{\pm}\chi_{\mu}^{\mp}(z)s_{\mu}(z). \tag{3.1}$$

Let us now multiply both sides of (3.1) by $s_{\lambda}(z)$ and notice that

$$z\int_0^z s_{\mu}(\rho)s_{\lambda}(\rho)\rho^{-2}\,d\rho = (\lambda+\mu)^{-1}s_{\mu}(z)s_{\lambda}(z). \tag{3.2}$$

We obtain the integral formulas, valid for $\mathrm{Re}\,\lambda > -\eta$

$$\chi_{\lambda}^{\pm}(z) = F^{\pm}(z)s_{\lambda}(z) - \int_0^z K^{\pm}(z,\zeta)s_{\lambda}(\zeta)\zeta^{-2}\,d\zeta, \tag{3.3}$$

where

$$K^{\pm}(z,z') = \sum_{\mu \in S}\gamma_{\mu}(W^{\pm}, 1)\chi_{\mu}^{\mp}(z)s_{\mu}(z'). \tag{3.4}$$

[25] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (The Clarendon Press, Oxford, 1965), p. 263.

Let us now give to λ successively all the values ν belonging to the set S; multiply, for every ν, both sides of (3.3) by $\gamma_\nu(W^\pm, 1)s_\nu(z')$; and sum all the terms. We obtain

$$K^\pm(z\ z') = F^\mp(z)f^\pm(z, z') - \int_0^z K^\mp(z, \zeta)f^\pm(\zeta, z')\zeta^{-2}\, d\zeta, \tag{3.5}$$

where

$$f^\pm(z, z') = \sum_{\mu\in S}\gamma_\mu(W^\pm, 1)s_\mu(z)s_\mu(z'). \tag{3.6}$$

It follows from (3.3) that

$$s_0(z)[F^+(z) - F^-(z)] = \int_0^z [K^+(z, \zeta) - K^-(z, \zeta)]s_0(\zeta)\zeta^{-2}\, d\zeta \tag{3.7}$$

and from (2.35) that

$$F^+(z)F^-(z) = 1. \tag{3.8}$$

The system of equations (3.5), (3.7), and (3.8) replaces the Regge–Newton equation for the scattering of a spinning particle. It provides a formal way to derive from two input functions $f^+(z, z')$ and $f^-(z, z')$ [defined by (3.6) and the data of the γ_μ's] the potentials $\mathbf{U}(z)$ and $Q(z)$, and to construct all the solutions of Schrödinger equation. Existence and uniqueness of the solutions of this system will be studied below (Sec. 3C).

Another Integral Formula: Generating Functions

From the unsubtracted interpolation formula (2.36), in the same way as above, we can obtain

$$\chi_\lambda^\pm(z) = \lambda\int_0^z L^\pm(z, \zeta)s_\lambda(\zeta)\zeta^{-2}\, d\zeta \quad (\mathrm{Re}\,\lambda > 0), \tag{3.9}$$

where

$$L^\pm(z, z') = 2\pi^{-1}\sum_{\mu\in S^*} a_\mu^\pm \chi_\mu^\mp(z)s_\mu(z'). \tag{3.10}$$

Notice that, according to (2.35),

$$L^\pm(z, z) = zF^\pm(z). \tag{3.11}$$

It is easy to derive (3.1) from (3.9) by one integration by parts, after noticing that $s_\lambda(z)$ obeys the following differential equation:

$$T_0(z)s_\lambda(z) = \lambda s_\lambda(z), \tag{3.12}$$

where

$$T_0(z) = z(d/dz) - \tfrac{1}{2} = z^{\frac{3}{2}}(d/dz)z^{-\frac{1}{2}}. \tag{3.13}$$

If we refer to (2.21), which shows that $s_\mu(z)$ is essentially $z^{\mu+\frac{1}{2}}$, we see that $L^\pm(z, z')$ is a generating function for the $\chi_\lambda^\pm(z)$, which are obtained from it by a simple Mellin transform. We made the same remark in our previous paper[12] for λ-independent potentials.

B. Integral Equations on a General Base

The interpolation formulas (2.65) use as a base the regular wavefunctions corresponding to a certain λ-independent potential V. If we recall the Wronskian formula

$$\cdot\ \int_0^z \psi_\mu(\rho)\psi_\lambda(\rho)\rho^{-2}\, d\rho = \frac{\psi_\mu(z)\psi_\lambda'(z) - \psi_\lambda(z)\psi_\mu'(z)}{\lambda^2 - \mu^2}, \tag{3.14}$$

we see that (2.65) is equivalent to the following integral formula, valid for $\mathrm{Re}\,\lambda > -\epsilon$:

$$\chi_\lambda^\pm(z) = F^\pm(z)\psi_\lambda(z) - \int_0^z \mathcal{K}^\pm(z, \zeta)\psi_\lambda(\zeta)\zeta^{-2}\, d\zeta, \tag{3.15}$$

where

$$\mathcal{K}^\pm(z, z') = I^\pm(z, z') - J^\pm(z, z'), \tag{3.16}$$

$$I^\pm(z, z') = \sum_{\mu\in S}\gamma_\mu(W^\pm, 1)\chi_\mu^\mp(z)\psi_\mu(z'), \tag{3.17}$$

$$J^\pm(z, z') = \sum_{\mu\in S}\gamma_\mu(V, 1)\chi_\mu^\pm(z)\psi_\mu(z'). \tag{3.18}$$

Elementary algebraic manipulations enable us to derive from (3.15)–(3.18) the following system of integral equations:

$$J^\pm(z, z') = F^\pm(z)e(z, z') - \int_0^z \mathcal{K}^\pm(z, \zeta)e(\zeta, z')\zeta^{-2}\, d\zeta, \tag{3.19}$$

$$I^\mp(z, z') = F^\pm(z)g^\mp(z, z') - \int_0^z \mathcal{K}^\pm(z, \zeta)g^\mp(\zeta, z')\zeta^{-2}\, d\zeta \tag{3.20}$$

where

$$e(z, z') = \sum_{\mu\in S}\gamma_\mu(V, 1)\psi_\mu(z)\psi_\mu(z'), \tag{3.21}$$

$$g^\pm(z, z') = \sum_{\mu\in S}\gamma_\mu(W^\pm, 1)\psi_\mu(z)\psi_\mu(z'). \tag{3.22}$$

Now the following formula is derived straightforwardly from (3.15):

$$\psi_0(z)[F^+(z) - F^-(z)] = \int_0^z [\mathcal{K}^+(z, \zeta) - \mathcal{K}^-(z, \zeta)]\psi_0(\zeta)\zeta^{-2}\, d\zeta, \tag{3.23}$$

whereas (3.8) still holds.

The system of equations (3.16), (3.19), (3.20), (3.23), and (3.8) enables us, at least formally, to derive the potential and the regular wavefunctions from the input function $g^\pm(z, z')$ [since $e(z, z')$ is a known function]. The solutions will be studied below (Sec. 3D).

Another Integral Formula

From the unsubtracted interpolation formula (2.66) we easily derive the integral formula

$$\chi_\lambda^\pm(z) = \lambda\int_0^z \mathcal{L}^\pm(z, \zeta)\psi_\lambda(\zeta)\zeta^{-2}\, d\zeta, \tag{3.24}$$

valid only for $\operatorname{Re}\lambda > 0$, and where

$$\bar{L}^{\pm}(z, z') = 2\pi^{-1}\{\chi_0(z)\psi_0(z) + \sum_{\mu\in S}[b_\mu\chi_\mu^{\pm}(z) + a_{\mu}^{\pm}\chi_\mu^{\mp}(z)]\psi_\mu(z')\}. \quad (3.25)$$

C. Existence and Uniqueness of Solutions for Zero Base

Up to now, we have only proved that, being given potentials in class A, we are able to construct the integral equations systems given in Secs. 3A and 3B. In order to construct potentials and wavefunctions from two input functions $f^{\pm}(z, z')$ and $g^{\pm}(z, z')$, we have to show that it is possible to do so using those integral equations. Furthermore, we would like to be able to generate in this way potentials more general than those of class A.

Let us first discuss the existence of solutions of the zero base integral system, under the following assumptions:

Assumption A: $f^{\pm}(z, z')$ can be given the form

$$f^{\pm}(z, z') = (zz')^{\frac{1}{2}+\eta}\xi^{\pm}(z, z'), \quad (3.26a)$$

where η is a positive number, and it is possible to find a positive nondecreasing function $B(R)$, bounded for $R \le R_0 - \epsilon$, and such that

$$|\xi^{\pm}(z, z')| < B(R) \quad \text{for} \quad |z'| \le |z| \le R. \quad (3.26b)$$

Assumption A appears as a byproduct if we make the following assumption B, which is not necessary for discussing the solutions of the integral equations, but will be useful in the following.

Assumption B: $f^{\pm}(z, z')$ is given by the formula

$$f^{\pm}(z, z') = \int_E s_\mu(z)s_\mu(z')\, d[\alpha^{\pm}(\mu)], \quad (3.27)$$

where $\operatorname{Re}\mu \ge \eta$, $\forall \mu \in E$, and which converges uniformly for $|z'| \le |z| \le R_0 - \epsilon$. It is sufficient for this that

$$\int_E R_0^{2\mu}\frac{|d\alpha^{\pm}(\mu)|}{|\Gamma(1 + \mu)|^2} < C. \quad (3.28)$$

If $d[\alpha^{\pm}(\mu)]$ reduces to a set of δ functions at isolated points $\mu \in S$, (3.27) can be written as

$$f^{\pm}(z, z') = \sum_{\mu\in S}\gamma_\mu(W^{\pm}, 1)s_\mu(z)s_\mu(z') \quad (3.29)$$

and the input data are the γ_μ's as well as $f^{\pm}$.

Let us now set

$$K^{\pm}(z, z') = [F^+(z) + F^-(z)]k^{\pm}(z, z'), \quad (3.30)$$

$$\Delta(z) = [F^+(z) + F^-(z)]^{-1}[F^+(z) - F^-(z)]. \quad (3.31)$$

Substituting (3.30) and (3.31) in (3.5) and (3.7) leads us to the system of equations

$$k^+(z, z') = \tfrac{1}{2}[1 - \Delta(z)]f^+(z, z') - \int_0^z k^-(z, \zeta)f^+(\zeta, z')\zeta^{-2}\, d\zeta, \quad (3.32)$$

$$k^-(z, z') = \tfrac{1}{2}[1 + \Delta(z)]f^-(z, z') - \int_0^z k^+(z, \zeta)f^-(\zeta, z')\zeta^{-2}\, d\zeta, \quad (3.33)$$

$$\Delta(z) = z^{-\frac{1}{2}}\int_0^z [k^+(z, \zeta) - k^-(z, \zeta)]\zeta^{-\frac{3}{2}}\, d\zeta. \quad (3.34)$$

Equation (3.8) remains uncoupled with the system; (3.31) enables one to derive $F^+(z)$ and $F^-(z)$ from $\Delta(z)$ up to an arbitrary multiplicative function of z. The use of (3.8) is to ascertain this function.

Reduction of the System (3.32)–(3.34)

Let us multiply both sides of (3.32) by $f^-(z', z_1)$, both sides of (3.33) by $f^+(z', z_1)$, and integrate over z', with weight function z'^{-2}, on the interval $(0, z)$. Let us then replace the integrals in the right-hand sides by their values, given by (3.33) and (3.32), respectively. We obtain

$$k^-(z, z_1) = \tfrac{1}{2}[1 + \Delta(z)]f^-(z, z_1) - \tfrac{1}{2}[1 - \Delta(z)]\varphi_z^+(z, z_1) + \int_0^z k^-(z, \zeta)\varphi_z^+(\zeta, z_1)\zeta^{-2}\, d\zeta, \quad (3.35)$$

$$k^+(z, z_1) = \tfrac{1}{2}[1 - \Delta(z)]f^+(z, z_1) - \tfrac{1}{2}[1 + \Delta(z)]\varphi_z^-(z, z_1) + \int_0^z k^+(z, \zeta)\varphi_z^-(\zeta, z_1)\zeta^{-2}\, d\zeta, \quad (3.36)$$

where

$$\varphi_z^{\pm}(\zeta, z_1) = \int_0^z f^{\pm}(\zeta, z')f^{\mp}(z', z_1)z'^{-2}\, dz' = \varphi_z^{\mp}(z_1, \zeta). \quad (3.37)$$

Let us now introduce the resolvents $\Phi_z^{\pm}(x, y)$ associated to the kernels $\varphi_z^{\pm}(x, y)$ and the path of integration $(0, z)$, that is to say, the solutions of the Fredholm equations[26]:

$$\begin{aligned}\Phi_z^{\pm}(x, y) &= \varphi_z^{\pm}(x, y) + \int_0^z \Phi_z^{\pm}(x, t)\varphi_z^{\pm}(t, y)t^{-2}\, dt \\ &= \varphi_z^{\pm}(x, y) + \int_0^z \varphi_z^{\pm}(x, t)\Phi_z^{\pm}(t, y)t^{-2}\, dt \\ &= \Phi_z^{\mp}(y, x). \end{aligned} \quad (3.38)$$

[26] See, for example, F. G. Tricomi, *Integral Equations* (Interscience Publishers, Inc., New York, 1957).

Use of these resolvents enable us to write (3.35) and (3.36) as

$$k^-(z, z_1) = \tfrac{1}{2}\{[1 + \Delta(z)]\psi^-(z, z_1) - [1 - \Delta(z)]\Phi_z^+(z, z_1)\}, \quad (3.39)$$

$$k^+(z, z_1) = \tfrac{1}{2}\{[1 - \Delta(z)]\psi^+(z, z_1) - [1 + \Delta(z)]\Phi_z^-(z, z_1)\}, \quad (3.40)$$

where

$$\psi^\pm(z, z_1) = f^\pm(z, z_1) + \int_0^z f^\pm(z, t)\Phi_z^\mp(t, z_1)t^{-2}\, dt. \quad (3.41)$$

From (3.34), (3.39), and (3.40), we obtain the value of $\Delta(z)$:

$$\Delta(z) = \frac{z^{-\frac{1}{2}}\int_0^z \zeta^{-\frac{3}{2}}\, d\zeta\{\psi^+(z, \zeta) - \psi^-(z, \zeta) + \Phi_z^+(z, \zeta) - \Phi_z^-(z, \zeta)\}}{2 + z^{-\frac{1}{2}}\int_0^z \zeta^{-\frac{3}{2}}\, d\zeta\{\psi^+(z, \zeta) + \psi^-(z, \zeta) + \Phi_z^+(z, \zeta) + \Phi_z^-(z, \zeta)\}}. \quad (3.42)$$

Existence and Uniqueness of Solutions

The obtention of the resolvents $\Phi_z^\pm(x, y)$ is the key for the solution of the system (3.32)–(3.34). Besides, obtention of the potential from this solution seems to make necessary, according to (3.31) and (2.35), that $|\Delta(z)|$ be smaller than 1. However, we shall reassess this point in Sec. 3E below.

Let us now study the existence and uniqueness of the resolvents $\Phi_z^\pm$, that of all operators $k^\pm$, and give a domain in which $|\Delta(z)|$ is smaller than 1. All the proofs dealing with Fredholm equations are only sketched, since they are either trivial or very similar to those studied thoroughly in previous papers.[27]

Whether $f^\pm(r, r')$ obeys assumption A or assumption B, formulas (3.26) hold. As a result, provided $r < R$, $\varphi^\pm(r, r')$ belongs to $L_2(0, R)$ and its norm goes to zero like r,

$$N(\varphi^\pm, r) = \left[\int_0^r\int_0^r [\varphi^\pm(\rho, \rho')]^2\rho^{-2}\rho'^{-2}\, d\rho\, d\rho'\right]^{\frac{1}{2}} < \tfrac{1}{4}\eta^{-2}A^2(r)r^{4\eta}. \quad (3.43)$$

As a result, it is possible to find $r_0 > 0$ such that, for $r < r_0$, the Neuman series define a solution of (3.38). If $f(z, z')$ is an analytic function of $\zeta = z^{m^{-1}}$ and $\zeta' = (z')^{m^{-1}}$, this solution can be continued in the domain $|\zeta'| \le |\zeta| < r_0^{m^{-1}}$ as an analytic function of ζ and ζ'. Furthermore, $\Phi_z^\pm(x,y)$ can be given the form

$$\Phi_z^\pm(x, y) = (xy)^{\frac{1}{2}+\eta}A_z^\pm(x, y), \quad (3.44)$$

where $|A_z^\pm(x, y)|$ can be bounded by a nondecreasing function $\alpha(r)$, going to zero like $r^{2\eta}$ as r goes to zero, for any couple of complex points x and y lying in the circle centered at the origin and with radius r. It follows that $\Delta(z)$ goes to zero like $C\eta^{-1}z^{2\eta}$ as z goes to zero. It follows also that there is a domain $|\zeta'| \le |\zeta| < \omega$ in which the following conditions are simultaneously fulfilled:

(1) $\Delta(z)$ and $k^\pm(z, z')$ are analytic functions of ζ and ζ';

(2) $|\Delta(z)|$ is smaller than 1.

For $r > r_0$, it is possible to construct the resolvent, provided that $f^\pm(r, r')$, $(r' \le r)$, belongs to $L_2(0, r)$. If η is $\ge \frac{1}{2}$, the kernel in (3.38) is regular at the origin; and if it remains bounded in the domain with which we deal, the resolvent can be written down using the well-known Fredholm method.[26] If η is smaller than $\frac{1}{2}$ (but still positive), trivial changes of functions and variables enable one to reduce (3.38) to the Fredholm form with regular kernel, so as to be able to construct the resolvent. Outside of the real axis, the resolvent can be continued in the complex plane when, for instance, $f^\pm(z, z')$ is an analytic function of ζ and ζ'. The resolvent $\Phi^\pm(z, z')$ is then an analytic function of ζ and a meromorphic function of ζ. The solution of the Fredholm equation defined in this way is unique, except at the poles. Since these are isolated points, which, furthermore, do not exist in the circle where the Neuman series converges, and since we are looking for a solution which should be a continuous function of r, we can assert that the solution is unique in that sense.

D. Existence and Uniqueness of Solutions for any Base

In the general system G[(3.16), (3.19), (3.20), (3.23), and (3.8)], three functions are involved, $e(z, z')$ and $g^\pm(z, z')$. The first one is a known function, defined by (3.21), and therefore given with the data of the base $V(r)$. Assume, for example, that $V(r)$ belongs to the class A. The coefficients $\gamma_\mu(V, 1)$ are then easily obtained through the relation between $\psi_\mu(z)$ and $\psi_{-\mu}(z)$ $(\mu \in S)$, or between $\psi_\mu(z)$ and the residue of $\psi_{-\mu}(z)$ when this function happens to be infinite. Besides, in order to have a correct base, we should make sure that $\psi_\mu(z)$ is correctly defined and bounded throughout the domain we need. We

[27] See Sec. 1 of Refs. 6, 11, and 12.

therefore make the following assumptions, which appear as a byproduct when $V(r)$ has been constructed, either from the Bessel functions or from the functions $s_\mu(z)$, through the method recalled in Sec. 2A.

Assumption C: The base potential is chosen in such a way that the base functions $|\psi_\mu(z)|$ are bounded, for $\mathrm{Re}\,\mu \geq 0$ and for a domain which includes at least the circle $|z| < R_0$, by $C\,|\Gamma(1+\mu)|^{-1}\,|z|^{\mu+\frac{1}{2}}$. In this domain, the function $e(z, z')$, defined by the expansion (3.21), can be given the form

$$e(z, z') = (zz')^{\frac{1}{2}+\eta}\quad E(z, z'), \tag{3.45}$$

where $|E(z, z')|$ is bounded by a nondecreasing function $E(R)$, finite for $R < R_0$, provided that $|z'| \leq |z| \leq R$.

The functions $g^\pm(z, z')$ are the input functions. So as to be able to manage the integral equations, we only need that $g^\pm(z, z')$ fulfill the following assumption.

Assumption D: The functions $g^\pm(z, z')$ can be given the same form as $e(z, z')$ in (3.45), with the same bounds. It is needless to say that if the bounds or the domains are different from each other, we use the upper bound and the smaller domain.[28]

However, so as to be able to construct potentials, Assumption D is not sufficient, whereas the following Assumption E is.

Assumption E: The function $g^\pm(z, z')$ can be expanded on the base wavefunctions:

$$g^\pm(z, z') = \int_E \psi_\mu(z)\psi_\mu(z')\, d[g^\pm(\mu)], \tag{3.46}$$

where $\mathrm{Re}\,\mu \geq \eta > 0$, $\forall \mu \in E$. When $d[g^\pm(\mu)]$ reduces to a set of δ functions, we have the more convenient form

$$g^\pm(z, z') = \sum_S \psi_\mu(z)\psi_\mu(z')\gamma_\mu(W^\pm, 1). \tag{3.47}$$

Furthermore, we assume that the expansion of $g^\pm(z, z')$ converges uniformly for $|z'| \leq |z| \leq R_0 - \epsilon$. It is sufficient for this that (3.28) holds for the expansion coefficients.

There are two interesting ways of dealing with the system S. In the first one, for which we have to use Assumption E, we reduce S to an equivalent system with zero base. In the second one, we give a prominent importance to the difference between g^+ and g^-, and solve the system when $e(z, z')$ is chosen as the average of $g^+(z, z')$ and $g^-(z, z')$.

[28] Although we guess that the adjunction of differential conditions to Assumption D would give a set of assumptions sufficient for constructing potentials, the method we give here does not enable us to ensure this point.

Reduction to a Zero-Base System

We suppose that the base is either a constant potential base or a base obtained by the method recalled in Sec. 2. We suppose that $g^\pm(z, z')$ fulfills Assumption E. For the sake of convenience, we use the expansion (3.47). Extension of our results and proofs to the expansion (3.46) is straightforward. We have first to search solutions J and I of (3.19) and (3.20), where $K^\pm(z, z')$ is given by (3.16), whereas (3.23) and (3.8) hold.

Inserting (3.21) and (3.47) in (3.19) and (3.20) yields

$$I^\pm(z, z') = \sum_{\mu\in S} \gamma_\mu(W^\pm, 1)\ell_\mu^\mp(z)\psi_\mu(z'), \tag{3.48}$$

$$J(z, z') = \sum_{\mu\in S} \gamma_\mu(V, 1)\ell_\mu^\pm(z)\psi_\mu(z'), \tag{3.49}$$

where $\ell_\lambda^\pm(z)$ is *defined* by

$$\ell_\lambda^\pm(z) = F^\pm(z)\psi_\lambda(z) - \int_0^z \bar{K}^\pm(z, \zeta)\psi_\lambda(\zeta)\zeta^{-2}\, d\zeta. \tag{3.50}$$

Since the base has been constructed through our general method, we know[12] that it exists $K_1^V(z, z')$ such that

$$\psi_\lambda(z) = s_\lambda(z) - \int_0^z d\zeta\, \zeta^{-2} K_1^V(z, \zeta)s_\lambda(\zeta), \tag{3.51}$$

$$K_1^V(z, \zeta) = \sum \gamma_\mu(V, 1)\psi_\mu(z)s_\mu(\zeta) = -K_V^1(\zeta, z), \tag{3.52}$$

$$s_\lambda(z) = \psi_\lambda(z) - \int_0^z d\zeta\, \zeta^{-2} K_V^1(z, \zeta)\psi_\lambda(\zeta). \tag{3.53}$$

Insertion of (3.51) in (3.50) yields after some algebra

$$\begin{aligned} \ell_\lambda^\pm(z) = F^\pm(z)s_\lambda(z) &- \int_0^z d\zeta\, \zeta^{-2} s_\lambda(\zeta)\Big[F^\pm(z)K_1^V(z, \zeta) \\ &\qquad - \int_0^z \bar{K}^\pm(z, t)K_1^V(t, \zeta)t^{-2}\, dt\Big] \\ &- \int_0^z d\zeta\, \zeta^{-2} s_\lambda(\zeta)\Big[\bar{K}^\pm(z, \zeta) \\ &\qquad + \int_0^z \bar{K}^\pm(z, t)K_1^V(t, \zeta)t^{-2}\, dt\Big]. \end{aligned} \tag{3.54}$$

Insertion of (3.52) and use of (3.50) show that the first bracket in (3.54) is equal to

$$\sum \gamma_\mu(V, 1)s_\mu(\zeta)\ell_\mu^\pm(z).$$

Insertion of (3.16), (3.48), (3.49), and use of (3.53) and (3.52) show that the second bracket in (3.54) is equal to

$$\sum\, [\gamma_\mu(W^\pm, 1)\ell_\mu^\pm(z) - \gamma_\mu(V, 1)\ell_\mu^\pm(z)]s_\mu(\zeta).$$

We have therefore

$$\ell_\lambda^\pm(z) = F^\pm(z)s_\lambda(z) - \int_0^z K^\pm(z, \zeta)s_\lambda(\zeta)\zeta^{-2}\, d\zeta, \tag{3.55}$$

where

$$K^{\pm}(z\ \zeta) = \sum_{\mu} \gamma_\mu(W^{\pm}, 1)\mathscr{k}_\mu^{\mp}(z)s_\mu(\zeta), \qquad (3.56)$$

and this equation yields readily

$$K^{\pm}(z, z') = F^{\mp}(z)f^{\pm}(z, z') - \int_0^z K^{\mp}(z, \zeta)f^{\pm}(\zeta, z')\zeta^{-2}\, d\zeta, \qquad (3.57)$$

where

$$f^{\pm}(z, z') = \sum \gamma_\mu(W^{\pm}, 1)s_\mu(z)s_\mu(z'). \qquad (3.58)$$

We have now to transform (3.23), which can be written, using (3.51) and elementary transformations,

$$\begin{aligned} \psi_0(z)[F^+(z) - F^-(z)] &= \int_0^z s_0(\zeta)\zeta^{-2}\, d\zeta\Big[\mathcal{K}^+(z, \zeta) + \int_0^\zeta \mathcal{K}^+(z, \rho)K_1^V(\rho, \zeta)\rho^{-2}\, d\rho\Big] \\ &- \int_0^z s_0(\zeta)\zeta^{-2}\, d\zeta\Big[\mathcal{K}^-(z, \zeta) + \int_0^\zeta \mathcal{K}^-(z, \rho)K_1^V(\rho, \zeta)\rho^{-2}\, d\rho\Big] \\ &- \int_0^z s_0(\zeta)\zeta^{-2}\, d\zeta \int_0^z [\mathcal{K}^+(z, t) - \mathcal{K}^-(z, t)]K_1^V(t, \zeta)t^{-2}\, dt. \end{aligned} \qquad (3.59)$$

The same manipulations as used above for the right-hand side yield

$$\int_0^z s_0(\zeta)\zeta^{-2}\, d\zeta \sum \gamma_\mu(V, 1)s_\mu(\zeta)[\mathscr{k}_\mu^+(z) - \mathscr{k}_\mu^-(z)] - \int_0^z [F^+(z) - F^-(z)]K_1^V(z, \zeta)s_0(\zeta)\zeta^{-2}\, d\zeta.$$

According to (3.51), the last integral is equal to $[F^+(z) - F^-(z)](\psi_0(z) - s_0(z))$. Insertion of these results in (3.59) yields (3.7), with $K^{\pm}(z, z')$ defined by (3.56). Since Eq. (3.8) has the same form for any base, we have therefore proved that, being given $e(z, z')$ and $g^{\pm}(z, z')$, a system of equations identical to the zero base system associates $f^{\pm}(z, z')$ and $K^{\pm}(z, z')$; $f^{\pm}(z, z')$ is given from the input coefficients by (3.58). $K^{\pm}(z, z')$ is related to the "solution" $\mathcal{K}^{\pm}(z, z')$, which we are looking for, by (3.55) and the definitions (3.16), (3.48), and (3.49); the obtention of $K^{\pm}(z, z')$ enables one to obtain $\mathscr{k}_\lambda^{\pm}(z)$ through (3.55), and all the, functions can be calculated.

The existence and uniqueness of the solutions for any base are therefore related to the existence and uniqueness for the zero base. When Assumption E is fulfilled, they can be guaranteed, in the conditions and in the sense stated in Sec. 3C.

Direct Resolution of the System G

Let us set

$$\mathcal{K}^{\pm}(z, z') = [F^+(z) + F^-(z)]\mathscr{k}^{\pm}(z, z'). \qquad (3.60)$$

Using the notation $\Delta(z)$ defined by (3.31), rearranging (3.16), (3.19), and (3.20), and inserting (3.60) yields the equations

$$\begin{aligned} \mathscr{k}^{\pm}(z, z') &= \tfrac{1}{2}(1 \mp \Delta(z))g^{\pm}(z, z') - \tfrac{1}{2}(1 \pm \Delta(z))e(z, z') \\ &+ \int_0^z \mathscr{k}^{\pm}(z, \zeta)e(\zeta, z')\zeta^{-2}\, d\zeta \\ &- \int_0^z \mathscr{k}^{\mp}(z, \zeta)g^{\pm}(\zeta, z')\zeta^{-2}\, d\zeta, \end{aligned} \qquad (3.61)$$

$$\psi_0(z)\Delta(z) = \int_0^z [\mathscr{k}^+(z, \zeta) - \mathscr{k}^-(z, \zeta)]\psi_0(\zeta)\zeta^{-2}\, d\zeta. \qquad (3.62)$$

Equation (3.8) remains uncoupled with the previous equations. The solution of the system (3.61), (3.62) yields the functions $F^+(z)$ and $F^-(z)$ only up to a common multiplicative function $C(z)$, which should be ascertained by (3.8).

For the sake of simplicity let us assume that we have, in a first step, solved the λ-independent problem for an input function which is the average of g^+ and g^-. In other words, we choose the base so that $e(z, z')$ is equal to $\frac{1}{2}[g^+(z, z') + g^-(z, z')]$. If g^+ and g^- are given through their expansion coefficient, those of e are their arithmetic mean:

$$\gamma_\mu(V, 1) = \tfrac{1}{2}[\gamma_\mu(W^+, 1) + \gamma_\mu(W^-, 1)]. \qquad (3.63)$$

Let us set

$$\begin{aligned} g^{\pm}(z, z') &= e(z, z') \pm \delta(z, z'), \\ \mathscr{k}^{\pm}(z, z') &= k(z, z') \pm s(z, z'). \end{aligned} \qquad (3.64)$$

Substitution of (3.64) in (3.61) yields the equations

$$k(z, z') = -\tfrac{1}{2}\Delta(z)\delta(z, z') + \int_0^z s(z, \zeta)\delta(\zeta, z')\zeta^{-2}\, d\zeta, \qquad (3.65)$$

$$\begin{aligned} s(z, z') &= -\Delta(z)e(z, z') + \tfrac{1}{2}\delta(z, z') \\ &- \int_0^z k(z, \zeta)\delta(\zeta, z')\zeta^{-2}\, d\zeta \\ &+ 2\int_0^z s(z, \zeta)e(\zeta, z')\zeta^{-2}\, d\zeta. \end{aligned} \qquad (3.66)$$

Equation (3.65) yields readily $k(z, z')$ once $s(z, z')$ is known. Substitution of (3.65) in (3.66) yields

$$s(z, z') = \tfrac{1}{2}\Delta(z)h_z(z, z') + \tfrac{1}{2}\delta(z, z') - \int_0^z s(z, \zeta)h_z(\zeta, z')\zeta^{-2}\, d\zeta, \qquad (3.67)$$

where

$$h_z(x, y) = -2e(x, y) + \int_0^z \delta(x, \rho)\delta(\rho, y)\rho^{-2}\, d\rho. \qquad (3.68)$$

(2.67) is a Fredholm equation. Let $H_z(x, y)$, the resolvent associated to $h_z(x, y)$, be as follows:

$$\begin{aligned} H_z(x, y) &= h_z(x, y) - \int_0^z H_z(x, t)h_z(t, y)t^{-2}\, dt \\ &= h_z(x, y) - \int_0^z h_z(x, t)H_z(t, y)t^{-2}\, dt. \end{aligned} \qquad (3.69)$$

Use of (3.69) in (3.67) yields

$$s(z, z') = \tfrac{1}{2}\Delta(z)H_z(z, z') + \frac{1}{2}\left[\delta(z, z') - \int_0^z \delta(z, \zeta)H_z(\zeta, z')\zeta^{-2}\, d\zeta\right]. \quad (3.70)$$

Since it follows from (3.23) that

$$\psi_0(z)\Delta(z) = \int_0^z s(z, \zeta)\psi_0(\zeta)\zeta^{-2}\, d\zeta, \quad (3.71)$$

we get

$$\Delta(z) = \frac{\int_0^z \psi_0(\zeta)\zeta^{-2}\, d\zeta\left[\delta(z, \zeta) - \int_0^z \delta(z, \rho)H_z(\rho, \zeta)\rho^{-2}\, d\rho\right]}{2\psi_0(z) - \int_0^z H_z(z, \zeta)\psi_0(\zeta)\zeta^{-2}\, d\zeta}. \quad (3.72)$$

The formulas (3.72), (3.70), and (3.65) give the solution of the system (3.65), (3.66), and (3.71). This solution yields the solution of the system G only up to a multiplicative function, which should be defined by Eq. (3.8). The conditions of existence and uniqueness of the solutions are similar to those studied in Sec. 3C. Here again we can ensure the existence of a domain in which:

(a) the solution can be constructed;

(b) $|\Delta(z)|$ is smaller than 1 and goes to zero as z goes to zero;

(c) (3.8) ascertains the solution, which should be so that $F^{\pm}(z)$ goes to 1 as $z \to 0$.

Remark: It follows from the direct resolution of G that if $\delta(z, z')$ is equal to zero, $\bar{K}^{+}(z, z')$ and $\bar{K}^{-}(z, z')$ reduce to each other, whereas $F^{+}(z)$ and $F^{-}(z)$ reduce to 1. As a result, the system G reduces to

$$K(z, z') = f(z, z') - \int_0^z K(z, \zeta)f(\zeta, z')\zeta^{-2}\, d\zeta, \quad (3.73)$$

where

$$f(z, z') = g(z, z') - e(z, z'). \quad (3.74)$$

Equation (3.73) is nothing but the Regge–Newton equation—and $f(z, z')$ is defined, on the base ψ_μ, by its coefficients, say $\gamma_\mu(W, V)$. By making the reduction to a zero-base system, using the coefficients $\gamma_\mu(V, 1)$ and the corresponding resolvents, we obtain the zero-base integral equation, where $g(z, z')$ is defined by the coefficients $\gamma_\mu(W, V) + \gamma_\mu(V, 1)$. Now since in the zero-base system, by definition, the coefficients of $g(z, z')$ are the $\gamma_\mu(W, 1)$, we have there a proof of the linear formula (2.11) which is valid for all the classes of potentials that can be constructed through the method recalled in Sec. 2A.

E. Schrödinger Equation from the Integral Equations

Assumptions B and E are more general than those which define class A, since they include continuous distributions in the μ plane of real or complex expansions coefficients, whereas in class A we only set real coefficients defined for rational positive μ's. The assumptions A and D are still more general, but they are not sufficient for our purpose. However, if we compute[20] the assumption A with the following requirements, we obtain conditions sufficient for constructing potentials from the input functions $f^{\pm}(z, z')$ and the zero-base integral equations.

Assumption F: $f^{\pm}(z, z')$ obeys Assumption A and the following partial differential equation:

$$[T_0(x) - T_0(y)]f^{\pm}(x, y) = [D_0(x) - D_0(y)]f^{\pm}(x, y) = 0, \quad (3.75)$$

where $T_0(x)$ is given by (3.13), and

$$D_0(x) = T_0^2(x) = x^2(d^2/dx^2) + \tfrac{1}{4}. \quad (3.76)$$

Obviously, Assumption F is a byproduct if B is fulfilled.

Construction of the Potentials

We first give two remarks which will prove useful in the construction procedure.

Remark (1): According to the discussion of Sec. 3C, it is possible to find a nonvanishing domain Ω ($|z'| \le |z| \le R - \epsilon$), where a solution of the system (3.32), (3.33), and (3.34) exists and is unique, whereas $|\Delta(z)| < 1$. As a result, the system (3.5), (3.7) has a solution defined up to a multiplicative constant $C(z)$, which is ascertained by (3.8). This means that if $(F^{\pm}(z), K^{\pm}(z, z'))$ is the unique solution of the system (3.5), (3.7), and (3.8) in Ω, any solution $(K_c^{\pm}(z, z'), F_c^{\pm}(z))$ of the system (3.5), (3.7) is of the form

$$K_c^{\pm}(z, z') = C(z)K^{\pm}(z, z'), \quad (3.77a)$$

$$F_c^{\pm}(z) = C(z)F^{\pm}(z). \quad (3.77b)$$

Remark (2): It follows from Assumption A (or B) and the results of Sec. 2C that

$$\left.\begin{aligned} f^{\pm}(z, z') &\sim C(zz')^{\frac{1}{2}+\eta}, \\ K^{\pm}(z, z') &\sim C(zz')^{\frac{1}{2}+\eta}, \end{aligned}\right\} \quad \text{as} \quad z \text{ or } z' \to 0. \quad (3.78)$$

Construction Procedure

Let us introduce two functions $\bar{\mathbf{U}}(x)$ and $\bar{Q}(x)$, and let us set

$$\not{k}^{\pm}(z, z') = [D_0(z) + z^2\bar{\mathbf{U}}(z) - D_0(z') \pm 2z^2\bar{Q}(z)T_0(z')]K^{\pm}(z, z'), \quad (3.79)$$

where $K^{\pm}(z, z')$ is the solution of the system (3.5), (3.7), and (3.8).

It is a matter of straightforward and tedious differentiations, integration by parts, and use of (3.75) and (3.78) to show from (3.5a) and (3.5b) that

$$\not{k}^{\pm}(z, z') = \mathbf{E}^{\pm}(z, z') - \int_0^z \not{k}^{\mp}(z, \zeta)f^{\pm}(\zeta, z')\zeta^{-2}\, d\zeta, \quad (3.80)$$

$$z^{\frac{1}{2}}\mathbf{E}(z) = \int_0^z [\not{k}^{+}(z, \zeta) - \not{k}^{-}(z, \zeta)]\zeta^{-\frac{3}{2}}\, d\zeta, \quad (3.81)$$

where

$$\begin{aligned}\mathbf{E}^{\pm}&(z, z') \\ &= 2\left[z\frac{d}{dz}F^{\mp}(z) \pm z^2\bar{Q}(z)F^{\mp}(z)\right]z\frac{\partial}{\partial z}f^{\pm}(z, z') \\ &\quad + f^{\pm}(z, z')\left[z^2\mathbf{U}(z)F^{\mp}(z) - 2z\frac{d}{dz}z^{-1}K^{\mp}(z, z)\right. \\ &\quad \left. + z^2\frac{d^2}{dz^2}F^{\mp}(z) \mp 2z\bar{Q}(z)K^{\mp}(z, z) \mp z^2\bar{Q}(z)F^{\mp}(z)\right],\end{aligned} \quad (3.82)$$

$$\begin{aligned}\mathbf{E}(z) = &\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2\bar{\mathbf{U}}(z)\right)[F^{+}(z) - F^{-}(z)] \\ &- 2z\left[\left(\frac{d}{dz} - z\bar{Q}(z)\right)z^{-1}K^{+}(z, z)\right. \\ &\left. - \left(\frac{d}{dz} + z\bar{Q}(z)\right)z^{-1}K^{-}(z, z)\right].\end{aligned} \quad (3.83)$$

Let us now set $\bar{Q}(z)$ to be equal to $Q(z)$ defined in such a way that the coefficient of $(\partial/\partial z)f^{\pm}(z, z')$ in (3.82) is equal to zero. Since the equation (3.8) is fulfilled, the two following definitions of $Q(z)$ are consistent with each other:

$$zQ(z) = \pm[F^{\pm}(z)]^{-1}\frac{d}{dz}F^{\pm}(z). \quad (3.84)$$

Insertion of this result in (3.82) and insertion of (3.82) and (3.83) in (3.80) and (3.81) yield the system

$$\left.\begin{aligned}\not{k}^{\pm}(z, z') &= H^{\mp}(z)f^{\pm}(z, z') \\ &\quad - \int_0^z \not{k}^{\mp}(z, \zeta)f^{\pm}(\zeta, z')\zeta^{-2}\, d\zeta, \\ z^{\frac{1}{2}}[H^{+}(z) - H^{-}(z)] &= \int_0^z [\not{k}^{+}(z, \zeta) - \not{k}^{-}(z, \zeta)]\zeta^{-\frac{3}{2}}\, d\zeta,\end{aligned}\right\} \quad (3.85)$$

where

$$H^{\pm}(z) = \left[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2\bar{\mathbf{U}}(z)\right]F^{\pm}(z) - 2z\left[\frac{d}{dz} \mp zQ(z)\right]z^{-1}K^{\pm}(z, z). \quad (3.86)$$

According to (3.77), in any domain where (3.5), (3.7), and (3.8) have a unique solution, it follows from (3.86) that $\not{k}^{\pm}(z, z')$ and $H^{\pm}(z)$ are, respectively, equal to $k^{\pm}(z, z')$ and $F^{\pm}(z)$ times a function of z, say $C(\bar{\mathbf{U}}(z), z)$, when this function has been computed for the function $\bar{\mathbf{U}}(z)$. Let us then define a function $\mathbf{U}(z)$ by

$$z^2[\bar{\mathbf{U}}(z) - \mathbf{U}(z)] = C(\bar{\mathbf{U}}(z), z). \quad (3.87)$$

With this choice of $\bar{\mathbf{U}}(z)$, $H^{\pm}(z)$ is equal to zero; therefore the multiplicative function $C(\bar{\mathbf{U}}(z), z)$ is equal to zero, and therefore $\not{k}^{\pm}(z, z')$ is equal to zero. We have therefore proved that when the system (3.5), (3.7), and (3.8) has an unique solution, it is possible to construct a function $Q(z)$, defined *consistently by either* of the equations (3.84), and a function $U(z)$, defined *consistently by either* of the equations (3.88):

$$\left[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} + z^2\mathbf{U}(z)\right]F^{\pm}(z) - 2z\left[\frac{d}{dz} \mp zQ(z)\right]z^{-1}K^{\pm}(z, z) = 0. \quad (3.88)$$

Therefore

$$[D_0(z) + z^2\mathbf{U}(z) - D_0(z') \pm 2z^2Q(z)T_0(z')]K^{\pm}(z, z') = 0. \quad (3.89)$$

Schrödinger Equation

Let us now introduce the functions $\chi_\lambda^{\pm}(z)$, defined by the relation

$$\chi_\lambda^{\pm}(z) = F^{\pm}(z)s_\lambda(z) - \int_0^z K^{\pm}(z, \zeta)s_\lambda(\zeta)\zeta^{-2}\, d\zeta. \quad (3.90)$$

Here again, it is a matter of straightforward but tedious algebra to show that

$$[D_0(z) + z^2\mathbf{U}(z)]\chi_\lambda^{\pm}(z) = \lambda^2\chi_\lambda^{\pm}(z) \pm 2\lambda z^2Q(z)\chi_\lambda^{\pm}(z). \quad (3.91)$$

If Assumption B is fulfilled, substitution of (3.27) in (3.5) and use of (3.90) show that $K^{\pm}(z, z')$ is given by the expansion

$$K^{\pm}(z, z') = \int_E \chi_\mu^{\mp}(z)s_\mu(z')\, d[\alpha^{\pm}(\mu)]. \quad (3.92)$$

Insertion of (3.92) in (3.90) provides the Wronskian interpolation formula on the zero base:

$$\chi_\lambda^{\pm}(z) = F^{\pm}(z)s_\lambda(z) - \int_{\mu\in E} d[\alpha_\mu^{\pm}(z)]\chi_\mu^{\mp}(z)\int_0^z s_\mu(\rho)s_\lambda(\rho)\rho^{-2}\, d\rho. \quad (3.93)$$

General-Base System

For constructing potentials and solutions of the Schrödinger equation from the general-base system, we limit ourselves to the case in which Assumption E is fulfilled by the input functions $g^{\pm}(z, z')$. We can then use the method devised in Sec. 3D so as to reduce the initial system to a zero-base system, in which the input functions $f^{\pm}(z, z')$ fulfill Assumption B. This enables us to construct the potentials and the regular solutions of the Schrödinger equation, as done above.

Existence and Reality of the Potentials

In the course of our studies, we encountered three kinds of systems of integral equations:

(1) the *initial systems*, e.g., for the zero base, the system (3.5), (3.7), and (3.8);

(2) The *reduced systems*, in which we introduced the function $\Delta(z)$ defined by (3.31), and divided the unknown functions by convenient quantities, so as to obtain a system of equations in which $F^{\pm}(z)$ no longer appear—e.g., (3.32), (3.33), and (3.36);

(3) The *truncated systems*, which are the initial systems minus Eq. (3.8).

In Secs. 3C and 3D, we proved that if the input functions are analytic in a domain, any of the reduced systems has a unique solution, except perhaps at isolated values of z, for which the resolvents have poles.

With more general conditions, provided very weak assumptions are satisfied by the input functions (Assumptions A and D), *there is a domain* D in which the solution of a reduced system exists and is unique. Besides, D contains a domain $\Omega(|z'| \leq |z| < \omega)$, in which $|\Delta(z)|$ is smaller than 1, and goes to zero as z goes to zero. Let us denote by D_0 the connected part of D which contains Ω. As z and z' lie in D_0, we can use the input functions and the integral equations to continue the unknown functions of the reduced system in D_0. Let us now look at the relations between the reduced-systems and the general-systems solutions. As we know, any solution of the truncated system is of the form

$$\begin{aligned} {}^{T}K^{\pm}(z, z') &= C(z)k^{\pm}(z, z'), \\ {}^{T}F^{\pm}(z) &= C(z)\tfrac{1}{2}[1 \pm \Delta(z)], \end{aligned} \tag{3.94}$$

for the zero base, and similarly for the general base. The function $C(z)$ is ascertained by (3.8), which reads

$$\tfrac{1}{4}C^2(z)[1 - \Delta^2(z)] = 1, \tag{3.95}$$

and should be completed with the condition $C(O) = +2$, suppressing the ambiguity inside Ω. If z goes outside Ω, $\Delta^2(z)$ can be equal to 1. If this happens for an isolated value of z, it induces a pole for $\Delta^2(z)$. A cut may then be induced in the continuation (for instance, the analytic continuation) of $C(z)$—and therefore of the wavefunction. It is important to notice that since (3.84) and (3.88) are invariant when $F^{\pm}(z)$ and $K^{\pm}(z, z)$ are multiplied by a common constant factor, $\Delta^2(z)$ being equal to 1 does not induce a cut in the potentials. Actually both $Q(z)$ and $\mathbf{U}(z)$ can be given a rational expression in terms of $k^{\pm}(z, z)$, $\Delta(z)$, and their derivatives:

$$zQ(z) = [1 - \Delta^2(z)]^{-1}\Delta'(z), \tag{3.96}$$

$$\begin{aligned} z\mathbf{U}(z) = {} & \frac{4}{1 \pm \Delta}\frac{d}{dz}\, z^{-1}k^{\pm}(z, z) \\ & \mp 4\frac{\Delta'}{1 - \Delta^2}\frac{1 \mp \Delta}{1 \pm \Delta}\, z^{-1}k^{\pm}(z, z) \\ & \mp \frac{\Delta'}{1 - \Delta^2}\left[1 \pm \frac{z\Delta'}{1 - \Delta^2} + z\frac{\Delta''}{\Delta'} + 2z\frac{\Delta'\Delta}{1 - \Delta^2}\right]. \end{aligned} \tag{3.97}$$

Singularities are easy to study on these formulas. Suppose, for instance, that $f^{\pm}(z, z')$ are analytic functions in the domain we consider, so that the resolvent $\Phi_z(z, z')$ is a meromorphic function of z and an analytic function of z'. Then $k^{\pm}(z, z)$ can have but two kinds of singularities, viz., the poles of the resolvent and the poles of $\Delta(z)$, which is, in general, regular when the resolvent has a pole. Taking into account the linearity of $k^{\pm}(z, z)$ with respect to $\Delta(z)$, it is easy to see that the poles of $\Delta(z)$ do not give singularities in $zQ(z)$ and $\mathbf{U}(z)$. The poles of the resolvent give poles of order ≥ 2 to $z\mathbf{U}(z)$. Besides, when $(1 - \Delta^2)$ is equal to zero, $zQ(z)$ and $z\mathbf{U}(z)$ have poles, which are simple and double, respectively, if the zero is simple.

We shall not increase this study, which is, in our opinion, sufficient to give an idea of the care to be taken when constructing potentials with the present method.

F. Exactly Solvable Examples

Exactly solvable examples can be managed starting from the zero-energy base, using a few coefficients γ_μ. We give here, for convenience of the reader, the simplest one:

$$\left.\begin{aligned} f^{+}(z, z') &= 16\pi^{-1}\alpha s_1(z)s_1(z') = 2\alpha(zz')^{\frac{3}{2}}, \\ f^{-}(z, z') &= 0, \end{aligned}\right\} \tag{3.98}$$

which yields

$$
\begin{aligned}
Q(z) &= 2\alpha(1 + 2\alpha z^2)^{-1}, \\
\mathbf{U}(z) &= 4\alpha(1 - \alpha z^2)(1 + 2\alpha z^2)^{-2}, \\
\chi_\lambda^-(z) &= (1 + 2\alpha z^2)^{-\frac{1}{2}} s_\lambda(z), \\
\chi_\lambda^+(z) &= \left[(1 + 2\alpha z^2)^{\frac{1}{2}} \frac{\lambda}{\lambda + 1}\right. \\
&\quad \left. + (1 + 2\alpha z^2)^{-\frac{1}{2}} \frac{1}{\lambda + 1}\right] s_\lambda(z).
\end{aligned} \tag{3.99}
$$

It is pleasant to see in this example most of the singularities studied above. It is straightforward and tedious to verify the Schrödinger equation (3.91).

Correspondence with the Figure

The correspondence of the zero-base machinery with Fig. 1 is as follows. The input functions are $f^\pm(z, z')$. The properties (1) are either (3.26) and (3.75) or (3.27). The output generator is $K^\pm(z, z')$. The properties (2) are (3.89) and (3.78). Relations A are (3.5), (3.7), and (3.8). Relation B is (3.91). C is the set (3.88), (3.89). D is (3.3).

The correspondence of the general base machinery with the figure is as follows. The input functions are $g^\pm(z, z')$. The property (1) is (3.47). The output generator is $\bar{K}^\pm(z, z')$. The properties (2) are (3.16), (3.17), and (3.18). Relations A are (3.19), (3.20), (3.23), and (3.8), with reference to (3.17) and (3.18). Relation B is (3.91). C follows from (2.35), (2.57), and (2.58). D is (3.15).

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Extension of the Factorization Method

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(Received 13 September 1967)

We show that it is possible to extend the formalism of the factorization method for any displacement in the spectrum space of any second-order differential equation. Following this, we show that we can extend, at least formally, the formalism for some nth-order ordinary differential equations.

I. INTRODUCTION

It is well known that both electromagnetic and quantum theory lead to equations of the type

$$\frac{d^2y}{dx^2} + r(x, m)y + \lambda y = 0 \tag{1}$$

(or can be transformed to this form). An outstanding problem related to the solution of these equations is the problem of raising and lowering operators; i.e., assuming that we deal with the discrete spectrum of (1) so that we can label the solutions by $y(\lambda, m)$, then we want to find first-order differential operators which connect the solution $y(\lambda, m)$ with $y(\lambda, m + 1)$:

$$
\begin{aligned}
Ry(\lambda, m) &= \alpha(\lambda, m + 1)\, y(\lambda, m + 1), \\
Ly(\lambda, m) &= \alpha(\lambda, m)\, y(\lambda, m - 1).
\end{aligned} \tag{2}
$$

The well-known factorization method[1] sets

$$
\begin{aligned}
R &= k(x, m + 1) - \frac{d}{dx}, \\
L &= k(x, m) + \frac{d}{dx},
\end{aligned} \tag{3}
$$

and discusses under what conditions the following equations hold:

$$
\begin{aligned}
&\left[k(x, m + 1) - \frac{d}{dx}\right] y(\lambda, m) \\
&\qquad = [\lambda - L(m + 1)]^{\frac{1}{2}} y(\lambda, m + 1), \\
&\left[k(x, m) + \frac{d}{dx}\right] y(\lambda, m) \\
&\qquad = [\lambda - L(m)]^{\frac{1}{2}} y(\lambda, m - 1).
\end{aligned} \tag{4}
$$

[1] L. Infeld and T. E. Hull, Rev. Mod. Phys. 23, 21 (1950).

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$$
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\end{aligned} \tag{4}
$$

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The explicit form for the functions $k(x, m)$ and $L(m)$ is, of course, related to the particular equation (1) with which we are dealing.

Thus the factorization method equips us with a procedure for performing integer displacements on the m-spectrum line (assuming m real). This is sufficient as long as the interest is focused on the discrete spectrum of the above equations. However, in recent years use has been made of solutions of Eq. (1) in which m and λ assume continuous (real or complex) values. This is the case, e.g., in Regge-pole theory where one uses Legendre polynomials with continuous indices. In what follows we shall therefore discuss a generalization of the factorization method for arbitrary displacement Δm. One may look on this generalization as an analytic continuation of the operators R and L in (3) above.

II. METHOD

In order to gain generality we assume that we deal with the complex differential equation

$$\frac{d^2y(z)}{dz^2} + r(z, m)y(z) + \lambda y(z) = 0. \qquad (5)$$

As a first step of our extension, we find operators which displace $y(\lambda, m)$ to $y(\lambda, m + \Delta m)$, where Δm is assumed to be real, while m may be complex. We denote the operators that displace m from $y(\lambda, m)$ to $y(\lambda, m + \Delta m)$ by $H^{m,\Delta m}$[2] (Δm is a positive or negative real number, $|\Delta m| > 0$) and try to find a solution for $H^{m,\Delta m}$ of the form[3]

$$H^{m,\Delta m} = \left[k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) - \operatorname{sgn}(\Delta m)\frac{d}{dz}\right], \qquad (6)$$

so that

$$\left[k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) - \operatorname{sgn}(\Delta m)\frac{d}{dz}\right]y(\lambda, m) = \left[\lambda - L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)\right]^{\frac{1}{2}} y(\lambda, m + \Delta m). \qquad (7)$$

One notes immediately that when $\Delta m = \pm 1$, the operators (7) coincide in their form with those of (4).

We want to fix $k(z, m)$ so that

$$[H^{m+\Delta m, -\Delta m} H^{m,\Delta m}]y(\lambda, m) = \left[\lambda - L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)\right]y(\lambda, m) \qquad (8)$$

coincides with the original equation (5) for $y(\lambda, m)$,[4] i.e.,

$$\left[k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) - \operatorname{sgn}(-\Delta m)\frac{d}{dz}\right] \times \left[k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) - \operatorname{sgn}(\Delta m)\frac{d}{dz}\right]y(\lambda, m) = \left[\lambda - L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)\right]y(\lambda, m). \qquad (9)$$

Expanding, we get

$$\left[k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)^2 - \operatorname{sgn}(-\Delta m) \times \frac{d}{dz}k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) - \frac{d^2}{dz^2}\right]y(\lambda, m) = \left[\lambda - L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)\right]y(\lambda, m). \qquad (10)$$

Equating (10) with (5), we get

$$k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)^2 - \operatorname{sgn}(-\Delta m) \times \frac{d}{dz}k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) + L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) = -r(z, m). \qquad (11)$$

If $\Delta m > 0$, we then get

$$k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)^2 + \frac{d}{dz}k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) + L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) = -r(z, m); \qquad (12)$$

while if $\Delta m < 0$, let us substitute $\Delta m' = -\Delta m$ and drop the prime after the substitution (so that in both cases now $\Delta m = |\Delta m|$):

$$k\left(z, m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) - \frac{d}{dz}k\left(z, m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) + L\left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) = -r(z, m). \qquad (13)$$

Subtracting (13) from (12), we get

$$k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)^2 - k\left(z, m + \tfrac{1}{2} - \frac{\Delta m}{2}\right)^2 + \frac{d}{dz}k\left(z, m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) + \frac{d}{dz}k\left(z, m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) = L\left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) - L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right). \qquad (14)$$

[2] We note that we may restrict ourselves to $|\Delta m| < 1$, since for $\Delta m > 1$ we may perform the displacement in steps.

[3] The form of the operators $H^{m,\Delta m}$ and L is not arbitrary. In fact, if we try to write

$$H^{m,\Delta m} = k[z, m + f(\Delta m)] - g(\Delta m)\frac{d}{dz},$$
$$L(m, \Delta m) = L[m + h(\Delta m)],$$

then it is easy to show that the only form that might work is the form chosen by us.

[4] This requirement is needed since one might look on Eq. (7) as a definition of $y(\lambda, m + \Delta m)$. Equation (9) assures us that $y(\lambda, m + \Delta m)$ satisfies Eq. (5) for $(\lambda, m + \Delta m)$.

Let us assume now that[5]

$$k(z, m, \Delta m) = k_0(z, \Delta m) + \left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) k_1(z, \Delta m). \quad (15)$$

[This form of $k(z, m, \Delta m)$ is compatible with the above explicit form of $k(z, m + \frac{1}{2} + \Delta m/2)$ since Δm is always positive in our notation.] Substituting (15) into (14), we get

$$\left[\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) k_1 + k_0\right]^2 - \left[k_0 + \left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) k_1\right]^2 + k_0' + \left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) k_1' + k_0' + \left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) k_1' = L\left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) - L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right). \quad (16)$$

Note now that the coefficient of k_1' is

$$\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) + \left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) = \left[\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)^2 - \left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right)^2\right] \frac{1}{\Delta m},$$

while the coefficient of k_0' is

$$2 = \left[2\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) - 2\left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right)\right] \frac{1}{\Delta m}.$$

Thus we may rewrite (16) as

$$\left\{\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)^2 \left[k_1^2 + \frac{1}{\Delta m} k_0'\right] + 2\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)\left[k_0 k_1 + \frac{1}{\Delta m} k_0'\right]\right\} - \left\{\left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right)^2 \left[k_1^2 + \frac{1}{\Delta m} k_1'\right] + 2\left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right)\left[k_0 k_1 + \frac{1}{\Delta m} k_0'\right]\right\} = L\left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right) - L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right). \quad (17)$$

Replacing z by $x = z\Delta m$ and denoting the derivative of k with respect to x by $\dot{k}$, we get for the general solution of Eq. (17)

$$L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) = -\left\{\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)^2 [k_1^2 + \dot{k}_1] + 2\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right)[k_0 k_1 + \dot{k}_0]\right\}. \quad (18)$$

This must hold for all values of m and therefore

$$k_1^2 + \dot{k}_1 = a, \quad (19)$$

$$k_0 k_1 + \dot{k}_0 = \begin{cases} -ca & a \neq 0, \\ b & a = 0, \end{cases} \quad (20)$$

where a, b, and c are independent of z, m, and Δm. The form of Eqs. (19) and (20) is identical with that of Eqs. (3.15) in Ref. 1.

Let us now turn to the case of imaginary displacement of the spectrum $i\Delta m$ (where Δm is real).

We denote once again the operators that displace from $y(\lambda, m)$ to $y(\lambda, m + i\Delta m)$ by $H^{m, i\Delta m}(|\Delta m| > 0)$, and try to find a solution of the form

$$H^{m,i\Delta m} = \left[k\left(z, m + \frac{i}{2} + \frac{i\Delta m}{2}\right) - \operatorname{sgn}(\Delta m) \frac{d}{dz}\right], \quad (21)$$

so that

$$\left[k\left(z, m + \frac{i}{2} + \frac{i\Delta m}{2}\right) - \operatorname{sgn}(\Delta m) \frac{d}{dz}\right] y(\lambda, m) = \left[\lambda - L\left(m + \frac{i}{2} + \frac{i\Delta m}{2}\right)\right]^{\frac{1}{2}} y(\lambda, m + i\Delta m). \quad (22)$$

By similar reasoning, we are led to the equation

$$k\left(z, m + \frac{i}{2} + \frac{i\Delta m}{2}\right)^2 - k\left(z, m + \frac{i}{2} - \frac{i\Delta m}{2}\right)^2 + \frac{d}{dz} k\left(z, m + \frac{i}{2} + \frac{i\Delta m}{2}\right) + \frac{d}{dz} k\left(z, m + \frac{i}{2} - \frac{i\Delta m}{2}\right) = L\left(m + \frac{i}{2} - \frac{i\Delta m}{2}\right) - L\left(m + \frac{i}{2} + \frac{i\Delta m}{2}\right). \quad (23)$$

Let us assume now that ($\Delta m > 0$)

$$k(z, m, i\Delta m) = k_0(z, i\Delta m) + \left(m + \frac{i}{2} + \frac{i\Delta m}{2}\right) k_1(z, i\Delta m). \quad (24)$$

[5] Here, and in other places, we shall not give the full formal justification of our steps. This has been done already in Ref. 1 and might be generalized easily in our case.

This leads to the equation

$$\left\{\left(m+\frac{i}{2}+\frac{i\Delta m}{2}\right)^2\left[k_1^2+\frac{1}{i\Delta m}k_1'\right]\right.$$
$$\left.+2\left(m+\frac{i}{2}+\frac{i\Delta m}{2}\right)\left[k_0k_1+\frac{1}{i\Delta m}k_0'\right]\right\}$$
$$-\left\{\left(m+\frac{i}{2}-\frac{i\Delta m}{2}\right)^2\left[k_1^2+\frac{1}{i\Delta m}k_1'\right]\right.$$
$$\left.+2\left(m+\frac{i}{2}-\frac{i\Delta m}{2}\right)\left[k_0k_1+\frac{1}{i\Delta m}k_0'\right]\right\}$$
$$=L\left(m+\frac{i}{2}-\frac{i\Delta m}{2}\right)-L\left(m+\frac{i}{2}+\frac{i\Delta m}{2}\right). \quad (25)$$

Substituting $x = iz\Delta m$ and denoting the derivative of k with respect to x by $\dot{k}$, we get for the general solution of (25)

$$L\left(m+\frac{i}{2}+\frac{i\Delta m}{2}\right)$$
$$=-\left\{\left(m+\frac{i}{2}+\frac{i\delta m}{2}\right)^2[k_1^2+\dot{k}_1]\right.$$
$$\left.+2\left(m+\frac{i}{2}+\frac{i\delta m}{2}\right)[k_0k_1+\dot{k}_0]\right\}, \quad (26)$$

which leads to Eqs. (19) and (20), and hence to the same solutions.

The same method described above is applicable to perform displacements on the λ plane. This can be done by solving the second-class problem (in the nomenclature of Ref. 1) of Eq. (5). We denote the operators that displace from (λ, m) to (λ, m') or (λ', m) by $H^{(\lambda,m),(\lambda,m')}$ and $H^{(\lambda,m),(\lambda',m)}$, respectively.

Until now we have dealt with operators in which the displacement Δm was different from zero. When Δm equals zero, it is natural to define $\operatorname{sgn}(0) = 0$ and $k(z, m, 0) = 1$ so that

$$H^{(\lambda,m),(\lambda,m)} = 1. \quad (27)$$

The above definitions are natural since zero is the mean of the jump in the values of the sign function at this point. On the other hand, since $k(z, m, \Delta m)$ depends on z through the form $z\Delta m$, it follows that k is independent of z when $\Delta m = 0$. Therefore, if we deal with normalized operators, we must fix $k(z, m, 0) = 1$.

Once we have found operators which displace the real and imaginary axis of the m plane, we may use these operators in order to perform displacements in an arbitrary direction. This can be done by multiplying two operators of the above kinds. Thus if we want to perform a displacement Δm such that

$$\Delta m = \operatorname{Re}\Delta m + i\operatorname{Im}\Delta m, \quad (28)$$

then the desired operator, which will be denoted by $D^{(\lambda,m),(\lambda,m+\Delta m)}$, is

$$D^{(\lambda,m),(\lambda,m+\Delta m)} = H^{(\lambda,m),(\lambda,m+\operatorname{Re}\Delta m)}H^{(\lambda,m),(\lambda,m+i\operatorname{Im}\Delta m)}. \quad (29)$$

However, once we find these displacement operators, we can perform other operations in the spectrum space. We illustrate the method for rotation operators in the real λm plane (other operators can be easily inferred).

To find these operators we use the polar-coordinate system $(r\theta)$ in the spectrum space:

$$\lambda = r\cos\theta, \quad m = r\sin\theta. \quad (30)$$

A rotation $(r, \theta) \rightarrow (r, \theta + \Delta\theta)$ means that we are looking for operators that displace from (λ, m) to (λ', m'), where

$$\lambda' = r\cos(\theta+\Delta\theta), \quad m' = r\sin(\theta+\Delta\theta). \quad (31)$$

Therefore the desired operator, which is denoted by $R^{(r,\theta),(r,\theta+\Delta\theta)}$, is

$$R^{(r,\theta),(r,\theta+\Delta\theta)} = D^{[r\cos\theta,r\sin(\theta+\Delta\theta)],[r\cos(\theta+\Delta\theta),r\sin(\theta+\Delta\theta)]}$$
$$\times D^{(r\cos\theta,r\sin\theta),[r\cos\theta,r\sin(\theta+\Delta\theta)]}. \quad (32)$$

We notice, however, that a rotation by $\Delta\theta$ may be accomplished in two ways: either by $\Delta\theta$ or $\Delta\theta - 2\pi$; the two operators do not coincide. This is due to the discontinuity in the passage from positive-displacement operators to negative ones. This discontinuity can be understood if we view our procedure as a method for calculating the square root of a differential operator, which leads, as usual, to a two-sheeted solution of the problem. In the following we may choose to work in either sheet as convenient.

In the above discussion we confine ourselves to ordinary second-order differential equations. Nevertheless, the whole discussion is valid for partial-differential equations which are separable. This is the case for the n-dimensional Laplacian. We observe, however, that in this case only two of the eigenvalues can be treated exactly as above. In order to displace other eigenvalues we must use combined first- and second-class operators. This follows from the fact that these eigenvalues appear in two ordinary equations which result from the separation of the partial-differential equation (see Sec. IV).

III. GLOBAL TRANSFORMATIONS IN THE SPECTRUM SPACE

Until now we have been dealing with operators in the z space which transform a specified function $y(\lambda, m)$ to another one $y(\lambda', m')$ [thus inducing a transformation in the spectrum space from (λ, m) to (λ', m')]. In this sense the operators found until now are point operators since they usually depend on the starting spectrum point (λ, m). We are looking now for operators which are global. This means that these operators will act on any function $y(\lambda, m)$ or a combination of these functions to produce a desired change in the dependence of these functions on the spectrum points. To this end we must define at first the space of functions upon which our global operators will operate.

At first let us observe that the integral

$$\int y(z, \lambda, m)^* y(z, \lambda', m')\, dz \tag{33}$$

for arbitrary (λ, m) might be divergent. Therefore we must use the technique of regularization[6] in order that (33) will converge and be zero for $(\lambda' m') \neq (\lambda, m)$. It was shown in Ref. 6 that this is possible if the functions $y(\lambda, m)$ are not too steep. Since our interest will be focused on spherical harmonics, this will be always possible. A detailed study of the regularization technique for the spherical harmonics is given in Refs. 7 and 8. So let us define

$$[y(\lambda, m), y(\lambda', m')] = \operatorname{reg} \int y(\lambda, m)^* y(\lambda', m') \rho(z)\, dz, \tag{34}$$

where $\rho(z)$ is a weighting function such that

$$[y(\lambda, m), y(\lambda' m')] = \alpha(\lambda, m) \delta(\lambda, \lambda') \delta(m, m'), \tag{35}$$

where α is not necessarily positive. However, we may assume that the y's are normalized so that $\alpha(\lambda, m) = \pm 1$. Denote those functions $y(\lambda, m)$ for which the bilinear form (35) defined above is positive [negative] by $y(\lambda, m, +)[y(\lambda, m, -)]$. In the following we deal with $\{y(\lambda, m, +)\}$, but $\{y(\lambda, m, -)\}$ can be treated exactly in the same way.

Let us now define the space of functions

$$S^+ = \left\{ f; f = \sum_{i=1}^{n} a^i y(\lambda_i, m_i, +), n \text{ finite} \right\}. \tag{36}$$

S^+ is a normal space in which one may define a scalar product in a natural way. According to a well-known theorem in the theory of Hilbert spaces, it is possible to embed such a space into a Hilbert space $\mathcal{H}^+$ so that S^+ is dense everywhere in $\mathcal{H}^+$. Therefore, for any $f \in \mathcal{H}^+$, it is possible to write

$$f = \sum_{\lambda} \sum_{m} a(\lambda, m) y(\lambda, m, +). \tag{37}$$

In a similar way, we can construct $\mathcal{H}^-$.

[6] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1964), Vol. I.

[7] J. Fischer and R. Raczka, International Centre for Theoretical Physics (Trieste) preprint IC/66/101.

[8] S. Sannikov, Yad. Fiz. **2**, 570 (1965) [Sov. J. Nuc. Phys. **2**, 407 (1966)].

In the space $\mathcal{H}^+ \oplus \mathcal{H}^-$, one may write, for any function,

$$f(z) = \iint d\lambda\, dm\, a(\lambda, m)\, y(z, \lambda, m), \tag{38}$$

where the above equality is strong (in the norm) and

$$\alpha(\lambda, m) = \operatorname{reg} \int f^*(z) y(z, \lambda, m) \rho(z)\, dz. \tag{39}$$

Having defined the space of functions and its topology, we will now write the global-transformation operators.

A. Translation Operators

Translations in the m plane:

$$ID^{\Delta m} = \iint d\lambda\, dm D^{(\lambda,m),(\lambda,m+\Delta m)} A_m^\lambda, \tag{40}$$

where

$$A_m^\lambda f = a(\lambda, m) y(\lambda, m); \tag{41}$$

in the λ plane we can write similarly

$$ID^{\Delta\lambda} = \iint d\lambda\, dm D^{(\lambda,m),(\lambda+\Delta\lambda,m)} A_m^\lambda, \tag{42}$$

and in a similar way we can write translation operators in any direction by combining λ- and m-translation operators.

B. Rotation Operators

We write down, as an example, the operators that rotate the real (λ, m) plane:

$$IR^{\Delta\theta} = \iint dr\, d\theta R^{(r,\theta)(r,\theta+\Delta\theta)} A_\theta^r, \tag{43}$$

where

$$A_\theta^r f = a(r, \theta) y(r, \theta). \tag{44}$$

In a similar way we may write other rotation operators.

The operators discussed above operate on the space $\mathcal{H}^+ \oplus \mathcal{H}^-$; however, each of these operators induces an appropriate transformation in the spectrum space. This is a result of the one-to-one correspondence between the points of the spectrum space and the functions $\{y(z, \lambda, m)\}$. Thus, if we denote by f the map

$$f: (\lambda, m) \to y(z, \lambda, m), \tag{45}$$

then the appropriate operator that displaces each point (λ, m) to $(\lambda, m + \Delta m)$ is $f^{-1} ID^{\Delta m} f$. In the following, we shall not write f explicitly and view the operators ID, IR, etc., as operators which operate on the spectrum space directly. In this way we are able to operate on the spectrum space by using operators which depend on the z-space coordinates. By means of the above procedures we can generate any regular transformation on the spectrum space. Thus we can express any transformation that belongs to $GL(2, c)$ on the spectrum space by means of z-space operators. It is easily seen that we can generalize our results to the n-dimensional case for which we get the group $GL(n, c)$.

IV. EXAMPLES

In the following, we give some examples in order to illustrate the method discussed above. We shall also deal with some algebraic aspects of our operators.

A. Associated Spherical Harmonics

Class I Problem

The differential equation is

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - \frac{m^2}{\sin^2\theta} + \lambda\right]P = 0. \quad (46)$$

We bring this equation to the standard form by means of the substitution

$$Y = \sin^{\frac{1}{2}}\theta P, \quad (47)$$

and the differential equation which results is

$$\frac{d^2Y}{d\theta^2} - \frac{(m^2 - \frac{1}{4})}{\sin^2\theta} Y + (\lambda + \tfrac{1}{4})Y = 0. \quad (48)$$

The solution we are looking for is given by

$$k(\theta, m, \Delta m) = \left(m + \frac{\Delta m}{2}\right)\cot(\theta\Delta m),$$

$$L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) = \left(m + \frac{\Delta m}{2}\right)^2. \quad (49)$$

Class II Problem

We introduce $l(l+1)$ for λ and replace $-m^2$ by λ. Equation (48) then becomes

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) + l(l+1) + \frac{\lambda}{\sin^2\theta}\right]P = 0. \quad (50)$$

By changing variables $z = \log\tan(\theta/2)$, we obtain for $\bar{P}(\theta)$

$$\frac{d^2\bar{P}}{dz^2} + \frac{l(l+1)}{\cosh^2 z}\bar{P} + \lambda\bar{P} = 0. \quad (51)$$

The solution is then

$$k(z, l, \Delta l) = \left(l + \tfrac{1}{2} + \frac{\Delta l}{2}\right)\tanh(z\Delta l),$$

$$L\left(l + \tfrac{1}{2} + \frac{\Delta l}{2}\right) = -\left(l + \tfrac{1}{2} + \frac{\Delta l}{2}\right)^2. \quad (52)$$

One can easily see now that for any $\Delta m > 0$ the operators

$$H^+ = \frac{1}{\Delta m}\int H^{m,\Delta m}A_m^\lambda\, d\lambda\, dm$$

and

$$H^- = \frac{1}{\Delta m}\int H^{m,-\Delta m}A_m^\lambda\, d\lambda\, dm, \quad (53)$$

with

$$L_0 = [H^+, H^-], \quad (54)$$

form an algebra which is isomorphic to $O(3)$. This can be easily seen by using the fact that

$$A_m^\lambda H^{m',\Delta m}A_m^{\lambda'} = \delta_{\lambda'}^{\lambda}\delta_{m'}^{m}H^{m',\Delta m}A_{m'}^{\lambda'}, \quad (55)$$

so that by (8) we get

$$\begin{aligned} L_0 = [H^+, H^-] &= \frac{1}{(\Delta m)^2}\int dm\, d\lambda \\ &\quad\times [H^{m-\Delta m,+\Delta m}H^{m,-\Delta m} - H^{m+\Delta m,-\Delta m}H^{m,\Delta m}]A_m^\lambda \\ &= \frac{1}{(\Delta m)^2}\int\left[L\left(m + \tfrac{1}{2} + \frac{\Delta m}{2}\right) - L\left(m + \tfrac{1}{2} - \frac{\Delta m}{2}\right)\right] \\ &\quad\times A_m^\lambda\, dm\, d\lambda = \frac{2}{\Delta m}\int mA_m^\lambda\, d\lambda\, dm \end{aligned} \quad (56)$$

and

$$\begin{aligned} [H^+, L_0] &= \int dm\, d\lambda\frac{H^{m,\Delta m}}{\Delta m}A_m^\lambda\int\frac{2m'}{\Delta m}A_{m'}^{\lambda'}\, dm'\, d\lambda' \\ &\quad - \int dm'\, d\lambda'\frac{2m'}{\Delta m}A_{m'}^{\lambda'}\int dm\, d\lambda, \end{aligned}$$

$$\begin{aligned} \frac{H^{m,\Delta m}}{\Delta m}A_m^\lambda &= \int dm\, d\lambda\left[\frac{2m}{\Delta m} - \frac{2(m+\Delta m)}{\Delta m}\right]\frac{H^{m,\Delta m}}{\Delta m}A_m^\lambda \\ &= -2H^+, \end{aligned} \quad (57)$$

while

$$[H^-, L_0] = 2H^-. \quad (58)$$

It is to be remarked that we could deal with the algebra $\{H^{\Delta m}\}_{\Delta m=-\infty}^{\infty}$. This algebra is an infinite-parameter Lie algebra. It plays no role in the following sections, although it may have some physical implications which will be dealt with elsewhere.

B. $O(3, 1)$ Spherical Harmonics

We deal with the following differential equation:

$$\begin{aligned} \Big\{-\frac{1}{\cosh^2\theta_1}\frac{\partial}{\partial\theta_1}\left(\cosh^2\theta_1\frac{\partial}{\partial\theta_1}\right) + \frac{1}{\cosh^2\theta_1}\frac{1}{\sin\theta_2} \\ \times\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right) - \frac{1}{\sin^2\theta_2}\frac{\partial^2}{\partial\theta_3^2}\Big\}P = \lambda P. \end{aligned} \quad (59)$$

In order to separate the equation, let us substitute

$$P = \psi(\theta_1)\varphi(\theta_2)\eta(\theta_3). \quad (60)$$

The differential equations which result are

$$\left\{\frac{1}{\cosh^2\theta_1}\frac{\partial}{\partial\theta_1}\left(\cosh^2\theta_1\frac{\partial}{\partial\theta_1}\right)+\frac{1}{\cosh^2\theta_1}l_2(l_2+1)\right\}\psi = l_1(l_1+2)\psi, \quad (61)$$

$$\left\{\frac{1}{\sin\theta_2}\frac{\partial}{\partial\theta_2}\left(\sin\theta_2\frac{\partial}{\partial\theta_2}\right)-\frac{m^2}{\sin^2\theta_2}\right\}\varphi = l_2(l_2+1)\varphi, \quad (62)$$

$$\frac{\partial^2}{\partial\theta_3^2}\eta(\theta_3) = -m^2\eta(\theta_3). \quad (63)$$

We see that Eqs. (62) and (63) are the same as for the associated spherical harmonics. Nevertheless, it is to be noticed that in order to perform a displacement in l_2 we must combine first-class operators [for (61)] with second-class operators [for (62)]. Class I operators for (61) are

$$k(\theta_1, l_1, \Delta l_2) = \left(l_2+\tfrac{1}{2}+\frac{\Delta l_2}{2}\right)\tanh(\theta_1\Delta\theta), \quad (64)$$

from which

$$L\left(l_2+\tfrac{1}{2}+\frac{\Delta l_2}{2}\right) = -\left(l_2+\tfrac{1}{2}+\frac{\Delta l_2}{2}\right)^2. \quad (65)$$

Class II operators are

$$k(\theta_1, l_1, \Delta l_1) = \left(l_1+\tfrac{3}{2}+\frac{\Delta l_1}{2}\right)\tan(\theta_1\Delta l_1), \quad (66)$$

from which

$$L\left(l_1+\tfrac{1}{2}+\frac{\Delta l_1}{2}\right) = \left(l_1+\tfrac{3}{2}+\frac{\Delta l_1}{2}\right)^2. \quad (67)$$

It has been shown by Raczka *et al.*[9] that the solutions of (59) for fixed λ provide a set of basis functions for the most degenerate representations of $O(3,1)$. Thus we see that one can build in the spectrum space of $O(3,1)$ the groups $GL(3, C)$ by space–time operators. We shall return to this point later.

V. INFINITESIMAL OPERATORS

Let us remark at first that a suitable set of matrices which build the algebra $GL(n, C)$ is given by

$$E_{nl} = (E_j^i)_{kl} = \delta_i^l\delta_k^j, \quad k, l = 1, \cdots, n, \quad (68)$$

and iE_{kl}, which we denote by E_{kl}^*. The commutation relation between these operators is then

$$\begin{aligned}[][E_{ij}, E_{kl}] &= -\delta_{kj}E_{il}+\delta_{il}E_{kj},\\ [E_{ij}, E_{kl}^*] &= 0, \\ [E_{ij}^*, E_{kl}^*] &= -(-\delta_{kj}E_{il}+\delta_{il}E_{kj}).\end{aligned} \quad (69)$$

It is to be noted that the n^2 operators $\{E_{ij}\}$ generate the algebra $U(n)$.

In order to build the desired operators, let us see the geometrical meaning of the operators E_{kl}. When we apply E_{kl} to a vector $\mathbf{x} = (x_1, \cdots, x_n)$, this operator destroys all components of $\mathbf{x}$ except the kth one, which is transformed into the lth component. The operators that we write down in order to build $GL(n, C)$ will have essentially the same meaning. In the following, we write down the infinitesimal operators for the algebras $GL(2, C)$ and $GL(3, C)$; the general case can be inferred easily from these two cases.

1. $GL(2, C)$. Let us define

$$E_{12} = \int dm\,\frac{H^{(0,0)(m,0)}}{\alpha}\frac{H^{(0,m)(0,0)}}{\beta}(A_m^0-\delta_0^l\delta_0^m), \quad (70)$$

$$E_{21} = \int dl\,\frac{H^{(0,0)(0,l)}}{\gamma}\frac{H^{(l,0)(0,0)}}{\delta}(A_0^l-\delta_0^l\delta_0^m), \quad (71)$$

$$H_1 = \int dl(A_0^l-\delta_0^l\delta_0^m), \quad (72)$$

$$H_2 = \int dm(A_m^0-\delta_0^l\delta_0^m), \quad (73)$$

where α, β, γ, δ are normalization constants. It is easy to verify that these operators satisfy the desired commutation relations.

2. $GL(3, C)$. In the same way as above we are able to define operators that satisfy the commutation relation of $GL(3, C)$. Thus E_{13} is given by

$$E_{13} = \int dl\,\frac{H^{(0,0,0)(0,0,l_1)}}{\alpha}\frac{H^{(l,0,0)}}{\beta}(A^{(l,0,0)}-\delta_0^{l_1}\delta_0^{l_2}\delta_0^{l_3}),$$

etc.

VI. METHOD OF EXTENSION FOR nth-ORDER EQUATIONS

The factorization method for second-order differential equations looks for first-order differential-operator solutions. However, it is clear that there exist other trivial solutions. These are second- and zero-order differential operators. Thus, we have for second-order differential equations three independent solutions. In the general nth-order eigenvalue problem,

$$\frac{d^n}{dx^n}y + r(x,m)y + \lambda y = 0. \quad (74)$$

We expect therefore to find $n+1$ independent solutions to our problem in the nth order, of which $n-1$ will be nontrivial.

The form which we impose on our operators is

$$\begin{aligned} R_m &= K(x, m+1) - D,\\ L_m &= K(x, m) + D,\end{aligned} \quad (75)$$

where

$$D = \frac{1}{\sqrt{2}}\left[\frac{d^i}{dx^i}+\frac{d^{n-i}}{dx^{n-i}}\right] \quad i = 1\cdots\left[\frac{n}{2}\right], \quad (76)$$

[9] R. Raczka, N. Limic, and J. Niederle, J. Math. Phys. 7, 1861 (1966).

so that

$$R_m y(\lambda, m) = [\lambda - L(m+1)]^{\frac{1}{2}} y(\lambda, m+1),$$
$$L_m y(\lambda, m) = [\lambda - L(m)]^{\frac{1}{2}} y(\lambda, m-1). \tag{77}$$

We say that Eq. (74) is factorized by R_m, L_m, if it can be replaced by each of the following two equations:

$$L_{m+1} R_m y(\lambda, m) = [\lambda - L(m+1)] y(\lambda, m), \tag{78}$$
$$R_{m-1} L_m y(\lambda, m) = [\lambda - L(m)] y(\lambda, m). \tag{79}$$

These equations lead to

$$\left\{k^2(x, m+1) + Dk(x, m+1) - \frac{d^n}{dx^n} - \frac{1}{2}\frac{d^{2i}}{dx^{2i}} - \frac{1}{2}\frac{d^{2(n-i)}}{dx^{2(n-i)}}\right\} y(\lambda, m) = [\lambda - L(m+1)] y(\lambda, m), \tag{80}$$

$$\left\{k^2(x, m) - Dk(x, m) - \frac{d^n}{dx^n} - \frac{1}{2}\frac{d^{2i}}{dx^{2i}} - \frac{1}{2}\frac{d^{2(n-i)}}{dx^{2(n-i)}}\right\} y(\lambda, m) = [\lambda - L(m)] y(\lambda, m). \tag{81}$$

Using Eq. (74), we are then led to

$$k^2(x, m+1) + Dk(x, m+1) - \frac{1}{2}\frac{d^{2i}}{dx^{2i}} - \frac{1}{2}\frac{d^{2(n-i)}}{dx^{2(n-i)}} + L(m+1) = -r(x, m), \tag{82}$$

$$k^2(x, m) - Dk(x, m) - \frac{1}{2}\frac{d^{2i}}{dx^{2i}} - \frac{1}{2}\frac{d^{2(n-i)}}{dx^{2(n-I)}} + L(m) = -r(x, m). \tag{83}$$

Subtracting (83) from (82), we then get

$$k^2(x, m+1) - k^2(x, m) + Dk(x, m-1) + Dk(x, m) = -L(m+1) + L(m). \tag{84}$$

Let us now substitute

$$k = k_0 + m k_1 \tag{85}$$

and denote

$$Dk_0 = k_0', \quad Dk_1 = k_1'.$$

We then have

$$[(m+1)^2(k_1^2 + k_1') + 2(m+1)(k_0 k_1 + k_0')] - [m^2(k_1^2 + k_1') + 2m(k_0 k_1 + k_0')] = L(m) - L(m+1). \tag{86}$$

The solution to this equation is

$$L(m) = -\{m^2(k_1^2 + k_1') + 2m(k_0 k_1 + k_0')\}, \tag{87}$$

since $L(m)$ is a function of m alone. We must then have

$$\frac{1}{2}\left[\frac{d^i}{dx^i} + \frac{d^{n-i}}{dx^{n-i}}\right] k_1 + k_1^2 = a^2, \tag{88}$$

$$\frac{1}{2}\left[\frac{d^i}{dx^i} + \frac{d^{n-i}}{dx^{n-i}}\right] k_0 + k_0 k_1 = \begin{cases} -ca & \text{if } a \neq 0, \\ b & \text{if } a = 0. \end{cases} \tag{89}$$

We shall deal with the solution of Eqs. (88) and (89) in the next section.

Until now we found only $[n/2]$ solutions to our factorization method; however, it is easy to infer the other solutions. These have the same form as in Eqs. (76), but in this case

$$D = \frac{i}{\sqrt{2}}\left[\frac{d^i}{dx^i} - \frac{d^{n-i}}{dx^{n-i}}\right].$$

The same procedure as in the preceding case follows. In the above discussion we dealt with integer displacements of the eigenfunctions of Eq. (74); however, it has been shown in Sec. I that the factorization method can be generalized to noninteger displacement (for second-order ordinary differential equations). The same methods apply in this more general case. We shall not dwell on the details.

VII. ON THE SOLUTION OF EQ. (88)

Equations (88) and (89) are the basic equations which we must solve in order that the procedure developed above will have a practical use.[10] However, this is not an easy task. The main difficulty lies in the fact that Eq. (88) is not linear. Thus it is possible to solve this equation in special cases only.

As an illustration, we deal with a fourth-order equation and take $i = n$, i.e., $i = 2$, so that

$$D = \frac{d^2}{dx^2} \tag{90}$$

(dropping unimportant numerical factors). Equation (88) then turns out to be

$$\frac{d^2}{dx^2} k_1 + k_1^2 = -a^2, \tag{91}$$

which lead to the solution

$$x = c_2 \pm \int (2c_1 - a^2 k_1 - \tfrac{1}{3}k^3)^{-\frac{1}{2}}\, dk.$$

This integral can be solved explicitly only for special values of a. For higher-order differential operators we face, of course, greater difficulties.

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[10] It is to be noticed that Eqs. (82) and (83) are operator equations. Therefore, even if we can solve Eq. (88), we can find the desired raising and lowering operators only if we impose on the solutions of (74) the subsidiary conditions

$$\left(\frac{d^{2i}}{dx^{2i}} + \frac{d^{2(n-i)}}{d^{2(n-i)}}\right) y = 0.$$

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Extended Energy-Integral Technique for Linear Differential Equations

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An extended energy-integral technique for boundary-value problems is presented for a class of differential equations which are prevalent in mathematical physics. The extended technique provides a method for answering questions in wave propagation and stability which could not be treated by the familiar method of energy integrals.

I. INTRODUCTION

In this paper, an extended energy-integral technique is derived for problems in linear differential equations. The Dirichlet–Neumann boundary-value problems are treated for equations in divergence form when the coefficients of the equation depend analytically on a parameter. These equations arise repeatedly in mathematical physics. They are especially prevalent in time-reduced problems.

Uniqueness theorems obtained from the familiar method of energy integrals rely heavily on the definiteness of the real or imaginary part of an integrand. However, when the coefficients of the differential equation depend on a parameter in a way which is not simple, the application of this technique in its usual form is often not possible.

In the extension of this technique all that is required here is a simple geometric constraint on coefficients of the differential equation. This constraint is only on the boundary of the domain of the parameter. It is independent of the dimension or extent of the domain of the independent variables of the equation.

The uniqueness theorems, which are proven in the next section, correspond to theorems which guarantee the absence of resonant frequencies in a complex domain of the parameter. The existence theorem, which is proven, corresponds to a theorem which guarantees the existence of a Laplace transform in a right half-plane of the transform variable.

While the treatment here is only given for second-order equations, obvious modifications will extend the technique to equations of higher order. Similarly, the technique employed here in the proof of the existence theorem on the whole space is readily extended to domains such as half-spaces, quadrants, etc. Existence on finite domains follows immediately from uniqueness.[1]

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[1] R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishing Co., Inc. N.Y. 1962), Vol. II.

II. UNIQUENESS

Consider the differential equation

$$\nabla_x \cdot \{a_1(x, z)\nabla_x u\} + a_2(x, z)u = -F(x, z). \quad (1)$$

Here ∇_x is the N-dimensional gradient in the variable $x = \{x_1, x_2, \cdots, x_N\}$. The domain of x is denoted by X and the boundary of X by ∂X. The functions $a_1(x, z)$ and $a_2(x, z)$ are taken to be continuous and single-valued in x and analytic in z for $z \in D$. It shall also be assumed that the $a_1(x, z)$ and $a_2(x, z)$ are continuous in z onto the boundary ∂D of D.

In the following, λ and μ shall always be taken to be real, nonzero numbers. For any continuous complex-valued function $f(z)$ defined on ∂D, the number W_f is defined to be the winding number, with respect to the origin of the complex plane, of the mapping of ∂D by $f(z)$. The class of functions $C_2^1(X)$ are those functions which have continuous first derivatives which are, together with those derivatives, square-integrable on X. H is the interior of a half-plane in the complex plane whose boundary ∂H contains the origin. The outward-drawn unit normal to ∂H is denoted by ν.

Three uniqueness theorems will be proved.

Theorem 1: Suppose that for each fixed $z \in \partial D$, there exists an $H(z)$, which is independent of x, λ, μ, and is such that

$$A_{\lambda\mu}(x, z) = \{-\lambda^2 a_1(x, z) + \mu^2 a_2(x, z)\} \in H(z).$$

Suppose it is possible to choose $H(z)$ so that $\nu(z)$ is Holder-continuous on ∂D, and that for such $\nu(z)$, $W_\nu = 0$. Then any solution in $C_2^1(X)$ of the Dirichlet–Neumann problems for Eq. (1) is unique for $z \in D$.

Theorem 2: Suppose that for each fixed $z \in \partial D$ there exists an $H(z)$, which is independent of x, λ, μ, and is such that $A_{\lambda\mu}(x, z) \in H(z)$. Suppose that with $\arg(-a_1) = b_1$ and $\arg(a_2) = b_2$, defined as continuous functions of z on ∂D,

$$\psi(z) = \tfrac{1}{2}[\max_{i,X} \{b_i\} + \min_{i,X} \{b_i\}]$$

is Holder-continuous and single-valued on ∂D. Then any solution in $C_2^1(X)$ of the Dirichlet–Neumann problems for Eq. (1) is unique for $z \in D$.

Theorem 3: Suppose that for $z \in \partial D$, there exists a real, single-valued Holder-continuous function $\gamma(z)$, and an H, independent of x, λ, μ, z, such that $e^{i\gamma(z)}A_{\lambda\mu}(x, z) \in H$. Then any solution in $C_2^1(X)$ of the Dirichlet–Neumann problem for Eq. (1) is unique.

Theorem 3 will be proved. It will then be shown that the hypotheses of all three theorems are equivalent.

Proof: Suppose that for some $z_0 \in D$ there exists two solutions in $C_2^1(X)$ to the Dirichlet–Neumann problem for Eq. (1). Then their difference $w(x, z_0)$ belongs to $C_2^1(X)$ and

$$\nabla_x \cdot \{a_1(x, z_0)\nabla_x w(x, z_0)\} + a_2(x, z_0)\, w(x, z_0) = 0, \quad (2)$$

together with appropriate data vanishing on ∂X. Multiplication by the complex conjugate of w, $\bar{w}(x, z_0)$, and integration by parts lead to

$$0 = \int_X dx\, A_{|\nabla_X w|, |w|}(x, z_0). \quad (3)$$

Thus,

$$E(z, z_0) = \int_X dx\{-a_1(x, z)\, |\nabla_x w(x, z_0)|^2 + a_2(x, z)\, |w(x, z_0)|^2\} \quad (4)$$

is analytic in z on D, with continuous extension to ∂D, and, by Eq. (3), has a zero at $z = z_0$.

Since $\gamma(z)$ is continuous on ∂D, there exists a regular real harmonic function $\xi(z)$ on D which takes the value $\gamma(z)$ on ∂D. There also exists a regular real harmonic function $\eta(z)$ such that $\varphi(z) = \xi(z) + i\eta(z)$ is analytic in D. If D is the half-plane $\operatorname{Im} z > 0$, and if $\lim \gamma(z) = 0$, then on ∂D[2]

$$\varphi(z) = \gamma(z) + \frac{1}{\pi i}\mathcal{P}\int_{\partial D} \frac{\gamma(\tau)}{\tau - z}\, d\tau.$$

Since $\gamma(\tau)$ is Holder-continuous, the principal-value integral is a Holder-continuous function of z on ∂D. Therefore, $\exp[i\varphi(z)]$ is an analytic function in D without zeros. Thus, the statement $E(z_0, z_0) = 0$ is equivalent to $0 = \exp[i\varphi(z_0)]R(z_0, z_0)$.

Let θ be the argument of the outward-drawn normal to H. Then for z on ∂D

$$\operatorname{Re}[e^{i(\theta+\varphi(z))}E(z, z_0)] < 0.$$

However, the argument principle of analytic function theory states that if $f(z)$ is analytic in D and continuous up to ∂D, then the number of zeros of $f(z)$ in D is precisely N_f.[3] In particular, if $f(\partial D) \subset H$, then $N_f = 0$ and $f(z) \neq 0$ for $z \in D$. Therefore, $E(z_0, z_0) \neq 0$. Thus $w(w, z_0) \equiv 0$.

Corollary: If a_1 and a_2 are not zero for any $x \in X$ and $z \in D$, then uniqueness under the same conditions is obtained for the equation

$$\nabla_x \cdot \left\{\frac{\nabla u}{a_1}\right\} + \frac{u}{a_2} = -F.$$

Proof: The proof follows by using

$$\int_X \left\{a_2(x, z)\left|\frac{w(x, z_0)}{a_2(x, z_0)}\right|^2 - a_1(x, z)\left|\frac{w(x, z_0)}{a_1(x, z_0)}\right|^2\right\} dx$$

in place of $E(z, z_0)$.

It will now be shown that the validity of the hypothesis of Theorem 1 implies the validity of the hypothesis of Theorem 2.

Proof: Suppose that the hypothesis of Theorem 1 is satisfied. Then the H with normals ν such that $\arg \nu(z) = \psi(z) + \pi$ are admissible. However, if $W_\nu = 0$, then $\psi(z)$ is a single-valued function of z on ∂D.

Proof: Similarly, the validity of the hypothesis of Theorem 2 implies that of Theorem 3.

This follows by taking $\gamma(z) = -(\psi(z))$. From the hypothesis of Theorem 2 we have

$$\psi(z) - \frac{\pi}{2} < \arg\{-\lambda^2 a_1(x, z) + \mu^2 a_2(x, z)\} < \psi(z) + \frac{\pi}{2}.$$

It follows that $\exp[-i\psi(z)]A_{\lambda\mu}(x, z) \in H$.

To complete the proof of equivalence of all three theorems, all that remains to be shown is that the hypothesis of Theorem 1 is a consequence of that of Theorem 3.

Proof: The existence of $\gamma(z)$ and H as given in Theorem 3 implies that there is at least one family $H(z)$ with normals $\nu(z)$ such that $W_\nu = 0$. What must be shown, therefore, is that W_ν is independent of the choice of $H(z)$ with Holder-continuous $\nu(z)$, provided $A_{\lambda\mu}(x, z) \in H(z)$.

Assume that there exists such a function $\nu^*(z)$ such that $W_{\nu^*} \neq 0$. Define on ∂D

$$P_\Gamma(z) = +e^{+i\Gamma(z)}\{-\lambda^2 a_1(x, z) + \mu^2 a_2(x, z)\} = e^{+i\Gamma(z)}A_{\lambda\mu}(x, z),$$

[2] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff, Ltd., Groningen, The Netherlands, 1953).

[3] K. Nopp, *Theory of Functions* (Dover Publications, Inc., New York, 1945), Part I.

with $W_{P_\Gamma} = 0$. Here, $\Gamma(z)$ is the analytic function with real part given by $\gamma(z)$ on ∂D. Therefore, $W_{A_{\lambda\mu}} = 0$. Thus, with

$$\phi(z) = \arg \nu^*(z) + \frac{1}{\pi i}\,\mathcal{P}\int_{\partial D} \frac{\arg \nu^*(\tau)}{\tau - z}\,d\tau$$

on ∂D, we have

$$0 = W_{e^{i\phi}A_{\lambda\mu}} = W_{e^{i\phi}}.$$

Hence, $\arg \nu^*(z)$ is single-valued on ∂D. This implies that $W_{\nu^*} = 0$.

III. EXISTENCE OF SOLUTIONS

In this section, the following existence theorem will be proven when $X \equiv R$, the whole x space.

Theorem 4: Assume that the conditions of Theorem 1 are met by the $a_i(x, z)$ of Eq. (1). Assume also that

$$\lim_{|x|\to\infty} a_i(x, z) = A_i(z), \quad i = 1, 2,$$

are analytic in D, and that $a_2(x, z)$ and $F(x, z)$ have two and that $a_1(x, z)$ has three square-integrable derivatives. If the functions $a_i(x, z) - A_i(z)$ and $F(x, z)$ are square-integrable, then there exists a square-integrable solution to Eq. (1) on R.

If, in addition,

$$\frac{\partial}{\partial z}(a_i - A_i), \quad \frac{\partial}{\partial z}(\nabla_x a_1), \quad \frac{\partial}{\partial z} a_2, \quad \text{and} \quad \frac{\partial}{\partial z} F$$

are square-integrable on R for $z \in D$, then the solution to Eq. (1) is analytic in z for $z \in D$.

Proof: Equation (1) is equivalent to

$$\begin{aligned} A_1\nabla_1^2 u + A_2 u &= -[F + \nabla_x \cdot \{(a_1 - A_1)\nabla_x u\} \\ &\quad + (a_2 - A_2)u] \\ &= -G. \end{aligned} \tag{5}$$

Treating the right-hand side of Eq. (5) as a known function and taking Fourier transforms of (5) yields

$$\hat{u}(k, z) = \frac{\hat{G}(k, z)}{|k|^2 A_1 - A_2}. \tag{6}$$

Here, k is the N-dimensional transform vector and $\hat{}$ denotes the Fourier transform. Since the hypothesis of Theorem 1 is satisfied by the $a_i(k, z)$, $|k|^2 A_1 - A_2 \neq 0$ for $z \in D$. Taking the inverse transform of Eq. (6) and integrating by parts to eliminate derivatives of $u(x, z)$ yields

$$u(x, z) = F_N^*(x, z) + \int_R K_N(x, y, z)u(y, z)\,dy, \tag{7}$$

with

$$a_1(x, z)F_N^*(x, z) = \frac{1}{(2\pi)^N}\int_R dk \int_R \frac{\exp[ik(x - y)]}{|k|^2 - A_2/A_1} F(y, z)\,dy.$$

Thus, $F_N^*(x, z) \in L^2(R)$. Also, suppressing the z dependence for simplicity,

$$a_1(x)K_N(x, y) = A_1\left[\nabla_y\left(\frac{a_1(y)}{A_1}\right) \cdot \nabla_y E_N + \left\{\frac{a_2(y)}{A_2} - \frac{a_1(y)}{A_1}\right\}\frac{A_2}{A_1} E_N\right],$$

$$E_N = \frac{1}{(2\pi)^N}\int_R dk \frac{\exp[ik(x - y)]}{|k|^2 - A_2/A_1}.$$

For $N = 1, 2, 3$, taking $\operatorname{Im}(A_2/A_1)^{\frac{1}{2}} > 0$,

$$E_1 = \frac{i}{2}\frac{\exp\{i[(A_2/A_1)^{\frac{1}{2}}]\,|x - y|\}}{(A_0/A_1)^{\frac{1}{2}}},$$

$$E_2 = \frac{1}{2\pi}\int_0^\infty \frac{\tau\,d\tau\,J_0(\tau)}{\tau^2 - (A_2/A_1)^{\frac{1}{2}}\,|x - y|^2}$$

(where $J_0(\tau)$ is the zeroth-order Bessel function which decays at infinity along the real axis), and

$$E_3 = \frac{1}{4\pi}\frac{\exp[i(A_2/A_1)^{\frac{1}{2}}\,|x - y|]}{|x - y|}.$$

In all of the above, $|x - y|$ is the usual Euclidean distance between the vectors x and y. Since the $a_i(x, z)$ satisfy the hypothesis of Theorem 4, the kernel K_N of Eq. (7) generates Fredholm integral equations. While for $N \geq 2$, the K_N are not square-integrable on $R \times R$, iterates of them are. This suffices for the application of the Fredholm alternative theorem.[4]

It will now be shown that any square-integrable solution of Eq. (7) is also a solution of Eq. (5). For simplicity, Eq. (7) may be rewritten in the form

$$\begin{aligned} a_1(x)u(x) &= \int_R dy\,F(y)H_0(|x - y|) \\ &\quad + \sum_{i=1}^{2}\int_R B_i(y)H_i(|x - y|)u(y)\,dy \\ &= \int_R d\xi\,F(\xi + x)H_0(|\xi|) \\ &\quad + \sum_{i=1}^{2}\int_R d\xi\,B_i(\xi + x)H_i(|\xi|)u(\xi + x), \end{aligned} \tag{8}$$

with the obvious identification of the H_0, B_i, and H_i with elements in Eq. (7).

Since the B_i and F have two derivatives as prescribed in the hypothesis of Theorem 4, it follows that, if $u(x, z)$ is a solution of Eq. (7) which is in $L^2(R)$, then its first and second derivatives exist and are in $L^2(R)$. For proof, one simply differentiates Eq. (8) with respect to x_j. The integral equation obtained for $\partial u/\partial x_j$ has the same kernel as that of Eq. (7). Only the inhomogeneity is different. It contains an integral of

[4] F. Tricomi, *Integral Equations* (Interscience Publishing Co., Inc., N.Y., 1957).

$u(x, z)$ and is also square-integrable. Similarly, employing again the smoothness of the $a_i(x, z)$, it is seen that $\partial^2 u/\partial x_j \partial x_k$ also exists and is in $L^2(R)$. In exactly the same way, if $u(x, z)$ is a solution of Eq. (7) and is in $L^2(R)$, $\partial u/\partial z$ exists and is in $L^2(R)$ for $z \in D$. Consequently, $u(x, z)$ is analytic in z for $z \in D$.

Since $u(x, z)$ has two derivatives, we may integrate by parts in Eq. (7) in order to eliminate singular behavior in $K_N(x, y)$. Thus, after integration by parts, we may differentiate with respect to x under the integral sign. Forming the combination of derivatives which appears on the left of Eq. (5), one readily finds that the derivatives of the integrals yield the combination which appears on the right of Eq. (5). Thus, any solution in $L^2(R)$ of Eq. (1) is also a solution of Eq. (7) and conversely.

However, the problem of finding a solution to Eq. (1) which is in $L^2(R)$ is equivalent to a boundary-value problem with a Dirichlet condition at infinity. By assumption, the hypothesis of Theorem 1 is satisfied. Therefore, any such solution is unique. The Fredholm alternative theorem, applied to Eq. (7), now implies that the solution exists.

IV. AN EXAMPLE

In this section, we illustrate the above technique of proving existence of solutions with an example. The equation we treat is of a type which appears in stability of ionized gases.[5] We include a treatment of it here for purposes of completeness. The equation has the advantage of having coefficients of a sufficiently complicated nature so as to discourage an analysis of it as a coupled system in its real and imaginary parts. We will assume that the coefficients which appear have the smoothness properties in x which are required in the hypothesis of Theorem 4. For our equation, we take

$$\nabla \cdot \{c_1(x)(z^2 - c_2(x))\nabla u\} + \left\{4\pi^{\frac{1}{2}}c_3(x) - \left(I_+(x, z) - \frac{I_-^2(x, z)}{I_+(x, z)}\right)\right\} \times u = -F(x, z), \quad (9)$$

with

$$I_\pm(x, z) = \int_{-\infty}^{+\infty} \frac{\tau\, d\tau}{\tau - z}\left\{n_1(x)\exp\left[-\frac{1}{2}\frac{\tau^2}{T_1^2(x)}\right] \pm n_2(x)\exp\left[-\frac{1}{2}\frac{\tau^2}{T_2^2(x)}\right]\right\} = \int_{-\infty}^{+\infty} \frac{\tau\, d\tau}{\tau - z} S_\pm(\tau, x).$$

We shall assume that the functions $n_j(x)$, $T_j^{-2}(x)$, and $c_k(x)$ are real, bounded, positive functions for $j = 1, 2$, and $k = 1, 2, 3$. We also assume that

$$c_3(x) > \frac{n_1(x)T_1(x)n_2(x)T_2(x)}{n_1(x)T_1(x) + n_2(x)T_2(x)} = p(x).$$

For $D(z)$, we take the half-plane $\operatorname{Im} z > 0$. The coefficients of the differential equation are analytic in this half-plane, and the functions $I_\pm(x, z)$ have a continuation onto the line $\operatorname{Im} z = 0$ which is given by

$$I_\pm(x, z) = \mathcal{P}\int_{-\infty}^{+\infty} \frac{\tau\, d\tau}{\tau - z} S_\pm(\tau, x) + \pi i z S_\pm(z, x) = R_\pm(z, x) + \pi i z S_\pm(z, x).$$

On the line $\operatorname{Im} z = 0$, the coefficient $a_1(x, z) = c_1(x)(z^2 - c_2(x))$ is real. The imaginary part of $a_2(x, z)$ on this line is given by

$$-\operatorname{Im}\left(I_+ - \frac{I_-^2}{I_+}\right) = \frac{\pi z}{R_+^2 + \pi^2 z^2 S_+^2}[S_+(|R_+|^2 - |R_-|)^2 + \pi^2 z^2 S_+(S_+^2 - S_-^2) + 2\{S_+(|R_+||R_-|) - S_-(R_+R_-)\}]$$

and therefore has the sign of $-z$ on the axis $\operatorname{Im} z = 0$. At $z = 0$, $a_1(x, z) = -c_1(x)c_2(x) < 0$ and $a_1(x, z) = 4\pi^{\frac{1}{2}}\{c_3(x) - p(x)\} > 0$. On any curve in the upper half z plane, whose distance from the origin is very much greater than unity, we have $a_1(x, z) = c_1(x)z^2 - c_1(x)c_2(x)$ and $a_2(x, z) \sim 4\pi^{\frac{1}{2}}c_3(x)$.

Therefore, on the boundary ∂D_σ of all such subdomains D_σ of the upper half z plane, the range of all combinations $-\lambda^2 a_1 + \mu^2 a_2$, with λ and μ as in Theorem 1, lies in the lower half of the complex plane if $\operatorname{Re} z > 0$ and $z \in \partial D_\sigma$. If, on the other hand, $\operatorname{Re} z < 0$ and $z \in \partial D_\sigma$, $-\lambda^2 a_1 + \mu^2 a_2$ belongs to the upper half of the complex plane. At $z = 0$, these combinations are all positive. They also have positive real part for large z when $\arg z$ is near $\pi/2$. Thus, for all such D_σ, we can choose complex half-planes $H(z)$ as in Theorem 1, with unit normals $\nu(z)$ such that the winding number of $\nu(z)$ on ∂D_σ with respect to the origin is zero. Since $D_\sigma(z)$ is arbitrarily large, we conclude from Theorem 4 that Eq. (9) has a unique solution which is in $L^2(R)$.

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[5] A. Kadish, Phys. Fluids **10**, 1614 (1967).

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Convergence of the Sudarshan Expansion for the Diagonal Coherent-State Weight Functional

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The mathematical properties of the original expansion derived by Sudarshan for the diagonal coherent-state weight functional are discussed. It is shown that, for stationary fields, the expansion is a generalized function in the space $Z'(R_2)$. The validity of this method of defining the weight functional in the case of arbitrary density operators and its relationship to other approaches to the problem of the diagonal representation is briefly considered.

I. INTRODUCTION

In recent years there has been considerable discussion of a "diagonal" representation of the density operator $\hat{\rho}$ which specifies the statistical state of a one-dimensional harmonic oscillator in terms of the right eigenstates of the boson annihilation operator, the so-called quasiclassical or coherent states $|z\rangle$.[1] This representation has the form of a superposition of the projection operators $|z\rangle\langle z|$ with weight functional $\varphi(z)$:

$$\hat{\rho} = \int \varphi(z)\,|z\rangle\langle z|\,d^2z. \tag{1}$$

Here $z = (x + iy)$, $d^2z = dx\,dy$ is the real element of area in the complex z plane, and the expansion of $|z\rangle$ in terms of the complete orthonormal set of harmonic-oscillator number eigenstates $|n\rangle$ $(n = 0, 1, 2, \cdots)$ is given by

$$|z\rangle = e^{-(|z|^2/2)} \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{\frac{1}{2}}}\,|n\rangle. \tag{2}$$

Using the representation (2), it is easy to verify that the coherent states are normalized to unity, nonorthogonal, and satisfy a completeness relation of the form[2]

$$\frac{1}{\pi}\int |z\rangle\langle z|\,d^2z = \hat{1}. \tag{3}$$

By virtue of (3) every density operator has the double-integral decomposition

$$\hat{\rho} = \frac{1}{\pi^2}\iint |z\rangle\langle z|\,\hat{\rho}\,|z'\rangle\langle z|\,d^2z\,d^2z', \tag{4}$$

where the integration is performed over the two complex variables z, z'. We note that the representations (1) and (4) are distinct; the diagonal form is *not* a degenerate case of the double-integral expansion which arises when the matrix element $\langle z|\,\hat{\rho}\,|z'\rangle$ is proportional to a delta function.[3]

The existence of a diagonal representation for the density operator characterizing a chaotic radiation field was noted by Glauber,[1] who has emphasized the central role of coherent-state expansions in the quantum theory of optical coherence.[4,5] The generality of the diagonal representation and the fact that its use leads to a symbolic equivalence between the classical and quantum formulations of coherence theory has been stressed by Sudarshan,[6] who derived an explicit expression for the weight functional $\varphi(z)$ which has the form of an infinite series of the Dirac delta function and its derivatives

$$\varphi(x) = \sum_{n=0}^{\infty} c_n \delta^{(n)}(x); \quad \delta^{(n)}(x) = \left(\frac{d}{dx}\right)^n \delta(x), \tag{5}$$

where the expansion coefficients c_n are proportional to the matrix elements of $\hat{\rho}$ in the number representation $\langle n|\,\hat{\rho}\,|m\rangle$. In the subsequent discussion of this result, it was shown[7] that when an infinite number of the c_n coefficients are nonzero, the series (5) does not converge in the generalized function space D', that is, it is not a distribution. The conclusion drawn by several authors[8,9] was that Eq. (5) is mathematically meaningless in this case, even in the context of distribution theory, and hence the symbolic equivalence which it defines is not valid. *This conclusion is misleading, however, since not every generalized function is a distribution.* In Sec. II of this paper we shall prove that for all density operators which are diagonal in the number representation and thus correspond to stationary fields, the explicit expression for the weight functional derived by Sudarshan, while not a distribution, does converge in the generalized function space Z'.[10] Since a unique diagonal

[1] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963).
[2] J. R. Klauder, Ann. Phys. (N.Y.) **11**, 123 (1960).
[3] M. M. Miller and E. A. Mishkin, Phys. Rev. **164**, 1610 (1967).
[4] R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[5] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).
[6] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).
[7] K. E. Cahill, Phys. Rev. **138**, B1566 (1965).
[8] D. Holliday and M. L. Sage, Phys. Rev. **138**, B485 (1965).
[9] R. Bonifacio, L. M. Narducci, and E. Montaldi, Nuovo Cimento **47**, 890 (1967).
[10] I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic Press Inc., New York, 1965), Vol. I, Chap. II.

representation in Z' exists for all density operators,[3] this result proves that the Sudarshan expansion for $\varphi(z)$ is mathematically meaningful for stationary fields, although other representations may be more useful in certain applications.[11] As a particular example of our result, we consider the diagonal representation of a thermal-radiation field. In Sec. III we consider the validity of the delta-function expansion for $\varphi(z)$ in the case of arbitrary density operators, and the relationship of this method of defining the weight functional to other approaches to the problem of the diagonal representation.

II. CONVERGENCE OF THE SUDARSHAN EXPANSION

For a single mode, Sudarshan's expansion for the weight functional has the form[6]

$$\varphi(z) = \sum_{n,m=0}^{\infty} \frac{\langle n|\hat{\rho}|m\rangle (n!\,m!)^{\frac{1}{2}}}{(n+m)!} \delta^{(n+m)}(r) \frac{e^{\{r^2+i(m-n)(\theta+\pi)\}}}{2\pi r}; \quad z = re^{i\theta}. \quad (6)$$

If the field is stationary, the density matrix is diagonal in the number representation

$$\langle n|\,\hat{\rho}\,|m\rangle = \langle n|\,\hat{\rho}\,|n\rangle\, \delta_{n,m}, \quad (7)$$

and Eq. (6) reduces to

$$\varphi(z) \to \varphi(r) = \sum_{n=0}^{\infty} \langle n|\,\hat{\rho}\,|n\rangle \frac{n!}{(2n)!} \delta^{(2n)}(r) \frac{e^{r^2}}{2\pi r}. \quad (8)$$

Considered as a function of two real variables, we wish to prove that this expression lies in the space $Z'(R_2)$, i.e., that Eq. (8) defines a continuous linear functional on the test-function space $Z(R_2)$, the dual space of $Z'(R_2)$. The spaces Z, Z' can be characterized in the following fashion: If f is an element in the space $Z(Z')$, then the Fourier transform of f, $\tilde{f}$, lies in $D(D')$, where the elements of D are infinitely differentiable functions of compact support. The Fourier transformation is a continuous, linear, one-to-one mapping between the spaces Z and D, Z' and D'. Although the spaces Z' and D' intersect, e.g., the space of tempered distributions S' is a common subspace, neither D' nor Z' is contained in the other. For this reason, functionals in Z' are sometimes called ultra-distributions.[12] We will first prove that Eq. (8) defines a functional on $Z(R_2)$ and then show that the functional is linear and continuous. Since the factor $2\pi r$ in the denominator of (8) is introduced to cancel the element of measure in the complex z plane, it suffices to consider the mapping

$$\langle\varphi(r), \psi(r,\theta)\rangle = \left\langle \sum_{n=0}^{\infty} \langle n|\,\hat{\rho}\,|n\rangle \frac{n!}{(2n)!} \delta^{(2n)}(r) e^{r^2}, \psi(r,\theta) \right\rangle. \quad (9)$$

Here $\langle\varphi, \psi\rangle$ denotes the complex number associated with each $\psi(r,\theta) \in Z(R_2)$. Using Leibnitz's rule for the derivative of a product, i.e.,

$$\left(\frac{d}{dx}\right)^n \{f(x)g(x)\} = \sum_{j=0}^{n} \binom{n}{j} \left\{\frac{d^j}{dx^j} f(x)\right\} \left\{\frac{d^{n-j}}{dx^{n-j}} h(x)\right\}, \quad (10)$$

where $\binom{n}{j} = \frac{n!}{j!\,(n-j)!}$ is the binomial coefficient, we can express the generalized function $\delta^{(2n)}(r)e^{r^2}$ as a finite sum of the delta function and its derivatives:

$$\begin{aligned}\delta^{(2n)}(r)e^{r^2}\psi(r,\theta) &= \left\{\frac{d^{2n}}{dr^{2n}} (e^{r^2}\psi(r,\theta))\right\}_{r=0} \\ &= \sum_{j=0}^{2n} \binom{2n}{j} \left\{\frac{d^j}{dr^j} e^{r^2}\right\}_{r=0} \left\{\frac{d^{(2n-j)}}{dr^{(2n-j)}} \psi(r,\theta)\right\}_{r=0}. \end{aligned} \quad (11)$$

Since

$$\left\{\frac{d^j}{dr^j} e^{r^2}\right\}_{r=0} = \begin{cases} \dfrac{j!}{\left(\frac{j}{2}\right)!}, & j \text{ even}, \\ 0, & j \text{ odd}, \end{cases} \quad (12)$$

then

$$\begin{aligned}\delta^{(2n)}(r)e^{r^2}\psi(r,\theta) &= \sum_{j=0}^{2n} \frac{(2n)!}{j!\,(2n-j)!} \frac{j!}{\left(\frac{j}{2}\right)!} \left\{\frac{d^{(2n-j)}}{dr^{(2n-j)}} \psi(r,\theta)\right\}_{r=0} \\ &= \sum_{k=0}^{n} \frac{(2n)!}{(2k)!\,(n-k)!} \psi^{(2k)}(0,\theta), \end{aligned} \quad (13)$$

where we have set $(2n-j) = 2k$. Substituting this result in Eq. (9), we obtain

$$\langle\varphi(r), \psi(r,\theta)\rangle = \sum_{n=0}^{\infty} \langle n|\hat{\rho}|n\rangle \sum_{k=0}^{n} \frac{n!}{(n-k)!\,(2k)!} \psi^{(2k)}(0,\theta). \quad (14)$$

In order to determine whether the double series (14) converges, it is necessary to obtain an explicit representation for $\psi^{(2k)}(0,\theta)$. Since $\psi(r,\theta)$ and $\tilde{\psi}(\rho,\gamma) \in D(R_2)$ are a Fourier transform pair

$$\psi(r,\theta) = \iint \tilde{\psi}(\rho,\gamma) e^{ir\rho\cos(\theta-\gamma)} \rho\, d\rho\, d\gamma, \quad (15)$$

then

$$\psi^{(2k)}(0,\theta) = \iint \tilde{\psi}(\rho,\gamma)(-1)^k \{\rho\cos(\theta-\gamma)\}^{2k} \rho\, d\rho\, d\gamma. \quad (16)$$

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[11] C. L. Mehta, Phys. Rev. Letters **18**, 752 (1967).

[12] J. Sebastiav e Silva, Ann. Math. **136**, 58 (1958).

Hence, from (14),

$$\langle\varphi(r), \psi(r,\theta)\rangle = \sum_{n=0}^{\infty}\langle n|\,\hat\rho\,|n\rangle \times \iint\left[\sum_{k=0}^{n}\frac{(-1)^k n!\{\rho\cos(\theta-\gamma)\}^{2k}}{(n-k)!\,(2k)!}\right]\tilde\psi(\rho,\gamma)\rho\,d\rho\,d\gamma. \quad (17)$$

Using the gamma-function duplication formula[13]

$$\Gamma(2z) = \frac{2^{2z-\frac12}}{(2\pi)^{\frac12}}\Gamma(z)\Gamma(z+\tfrac12), \quad (18)$$

we can reexpress the factor $(2k)!$

$$(2k)! = \Gamma(2k+1) = \frac{2^{2k}}{(\pi)^{\frac12}}k!\,\Gamma(k+\tfrac12) \quad (19)$$

and thereby reduce the finite sum in Eq. (17) to the standard form

$$\sum_{k=0}^{n}\frac{(-1)^k n!\,\{\rho\cos(\theta-\gamma)\}^{2k}}{(2k)!\,(n-k)!} = (\pi)^{\frac12}\sum_{k=0}^{n}\frac{(-1)^k n!\left\{\dfrac{\rho\cos(\theta-\gamma)}{2}\right\}^{2k}}{\Gamma(k+\frac12)(n-k)!\,k!} = M\left(-n,\tfrac12,\left\{\frac{\rho\cos(\theta-\gamma)}{2}\right\}^2\right), \quad (20)$$

where the confluent hypergeometric function $M(a,b,z)$ is defined by the infinite series[14]

$$M(a,b,z) = \sum_{k=0}^{\infty}\frac{\Gamma(a+k)\Gamma(b)}{\Gamma(a)\Gamma(b+k)}\frac{z^k}{k!}. \quad (21)$$

Note that when $a=-n$, $b=-m$, where n, m are integers, the infinite series reduces to a polynomial of degree n in z.

Now we can show that the series

$$\langle\varphi(r), \psi(r,\theta)\rangle = \sum_{n=0}^{\infty}\langle n|\,\hat\rho\,|n\rangle b_n, \quad (22)$$

where

$$b_n \equiv \iint M\left(-n,\tfrac12,\left\{\frac{\rho\cos(\theta-\gamma)}{2}\right\}^2\right)\tilde\psi(\rho,\gamma)\rho\,d\rho\,d\gamma, \quad (23)$$

is absolutely convergent. Let

$$a_n \equiv \langle n|\,\hat\rho\,|n\rangle\, b_n. \quad (24)$$

Then

$$|a_n| = \langle n|\,\hat\rho\,|n\rangle\,|b_n| \le \iint\left|M\left(-n,\tfrac12,\left\{\frac{\rho\cos(\theta-\gamma)}{2}\right\}^2\right)\right| \times |\tilde\psi(\rho,\gamma)|\,\rho\,d\rho\,d\gamma. \quad (25)$$

[13] M. Abramowitz and I. A. Stegun, Ed., *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1965), p. 256, formula (6.1.18).

[14] Reference 13, p. 504, formula (13.1.2).

Using the following asymptotic expansion for $M(a,b,z)$[15]:

$$M(a,b,z) = \Gamma(b)e^{z/2}(\tfrac12bz-az)^{\frac14-\frac12b} \times \cos\left\{(2bz-4az)^{\frac12}-\tfrac12b\pi+\frac{\pi}{4}\right\} \times\left[1+O\left\{\frac{1}{|\frac12b-a|^{\frac12}}\right\}\right], \quad (26)$$

as $a\to-\infty$, b bounded, z real, we obtain

$$\left|M\left(-n,\tfrac12,\left\{\frac{\rho\cos(\theta-\gamma)}{2}\right\}^2\right)\right| = (\pi)^{\frac12}\exp\left(\frac{\rho^2\cos^2(\theta-\gamma)}{8}\right)\left[1+O\left\{\frac{1}{(n+\frac14)^{\frac12}}\right\}\right] < C\exp\left(\frac{\rho^2\cos^2(\theta-\gamma)}{8}\right), \quad n>N, \quad (27)$$

where C is a constant.

Therefore

$$|a_n| < C\,\langle n|\,\hat\rho\,|n\rangle \times\iint\exp\left(\frac{\rho^2\cos^2(\theta-\gamma)}{8}\right)|\tilde\psi(\rho,\gamma)|\,\rho\,d\rho\,d\gamma = \langle n|\,\hat\rho\,|n\rangle\,C(\theta), \quad (28)$$

where $C(\theta)$ is a constant which is a function of the support of $\tilde\psi(\rho,\gamma)$ but is independent of n.

Since $\langle n|\,\hat\rho\,|n\rangle$ is summable, i.e.,

$$\sum_{n=0}^{\infty}\langle n|\,\hat\rho\,|n\rangle = 1, \quad (29)$$

it follows from Eqs. (28) that the series (22) is absolutely convergent; hence Eq. (8) defines a functional on $Z(R_2)$. The linearity of the functional follows directly from Eq. (14). For any two test functions $\psi_1(r,\theta)$, $\psi_2(r,\theta)\in Z(R_2)$ and complex constants c_1, c_2 we have

$$\langle\varphi(r), c_1\psi_1(r,\theta)+c_2\psi_2(r,\theta)\rangle = c_1\sum_{n=0}^{\infty}\langle n|\,\hat\rho\,|n\rangle\sum_{k=0}^{n}\frac{n!}{(n-k)!\,(2k)!}\psi_1^{(2k)}(0,\theta) + c_2\sum_{n=0}^{\infty}\langle n|\,\hat\rho\,|n\rangle\sum_{k=0}^{n}\frac{n!}{(n-k)!\,(2k)!}\psi_2^{(2k)}(0,\theta) = c_1\langle\varphi(r),\psi_1(r,\theta)\rangle + c_2\langle\varphi(r),\psi_2(r,\theta)\rangle. \quad (30)$$

To establish the continuity of the functional, we must show that if $\{\psi_v(r,\theta)\}_{v=1}^{\infty}$ is a sequence of test functions that converges to zero in $Z(R_2)$, then the number sequence $\{\langle\varphi(r),\tilde\psi_v(r,\theta)\rangle\}_{v=1}^{\infty}$ also converges to zero. We first note that the convergence to zero of the sequence $\{\psi_v(r,\theta)\}_{v=1}^{\infty}$ implies that the sequence of

[15] Reference 13, p. 508, formula (13.5.14).

Fourier transforms $\{\tilde{\psi}_v(\rho, \gamma)\}_{v=1}^{\infty}$ converges to zero in $D(R_2)$.[16] Since the latter functions have compact support, it follows that the numbers

$$\iint |\tilde{\psi}_v(\rho, \gamma)| \, \rho \, d\rho \, d\gamma \tag{31}$$

also converge to zero.

From Eq. (22),

$$|\langle \varphi(r), \psi_v(r, \theta) \rangle| \leq \sum_{n=0}^{\infty} \langle n | \hat{\rho} | n \rangle \iint \left| M\left(-n, \tfrac{1}{2}, \left\{\frac{\rho \cos(\theta - \gamma)}{2}\right\}^2\right) \right| \times |\tilde{\psi}_v(\rho, \gamma)| \, \rho \, d\rho \, d\gamma. \tag{32}$$

Using the relationship between $M(-n, \frac{1}{2}, x^2)$ and the Hermite polynomial of order $2n$, $H_{2n}(x)$[17]

$$M(-n, \tfrac{1}{2}, x^2) = (-1)^n \frac{n!}{(2n)!} H_{2n}(x), \tag{33}$$

and the following inequality satisfied by $H_{2n}(x)$[18]

$$|H_{2n}(x)| \leq k 2^n \{(2n)!\}^{\frac{1}{2}} e^{x^2/2}, \tag{34}$$

where $k \approx 1.086435$, we have

$$\left| M\left(-n, \tfrac{1}{2}, \left\{\frac{\rho \cos(\theta - \gamma)}{2}\right\}^2\right) \right| \leq k 2^n \frac{n!}{\{(2n)!\}^{\frac{1}{2}}} \exp\left(\frac{\rho^2 \cos^2(\theta - \gamma)}{8}\right). \tag{35}$$

Substituting this bound for

$$M\left(-n, \tfrac{1}{2}, \left\{\frac{\rho \cos(\theta - \gamma)}{2}\right\}^2\right)$$

for the first N terms of the sum in Eq. (32) and the asymptotic expansion Eq. (27) for the remaining terms, it follows that

$$|\langle \varphi(r), \psi_v(r, \theta) \rangle| < \left\{ \sum_{n=0}^{N} \frac{\langle n | \hat{\rho} | n \rangle k 2^n n!}{\{(2n)!\}^{\frac{1}{2}}} + C \sum_{n=N+1}^{\infty} \langle n | \hat{\rho} | n \rangle \right\} \times e^{R^2/8} \iint |\tilde{\psi}_v(\rho, \gamma)| \, \rho \, d\rho \, d\gamma, \tag{36}$$

where R is the maximum value of ρ. Since the first sum in Eq. (36) is finite and the infinite sum is known to be convergent, it follows from (31) that the right-hand side of (36) converges to zero as $v \to \infty$. This establishes the continuity of the functional and completes the proof of the convergence of the Sudarshan expansion for $\varphi(r)$ in $Z'(R_2)$.

As a particular example of this general result, we cite the diagonal representation of a radiation field in thermal equilibrium at temperature T. The density operator describing this statistical state is given by[19]

$$\hat{\rho} = \frac{e^{-\beta \hat{a}^+ \hat{a}}}{\mathrm{Tr}\, e^{-\beta \hat{a}^+ \hat{a}}} = e^{-\beta \hat{a}^+ \hat{a}}(1 - e^{-\beta}), \tag{37}$$

where $\hat{a}$, $\hat{a}^+$ are the boson annihilation and creation operators and $\beta = \hbar\omega/KT$. Substituting the number-representation matrix element of (37)

$$\langle n | \hat{\rho} | m \rangle = e^{-\beta n}(1 - e^{-\beta}) \delta_{n,m} \tag{38}$$

into Eq. (6), we obtain the Sudarshan expansion for the weight functional associated with a thermal-radiation field:

$$\varphi(r) = (1 - e^{-\beta}) \sum_{n=0}^{\infty} e^{-\beta n} \frac{n!}{(2n)!} \delta^{(2n)}(r) \frac{e^{r^2}}{2\pi r}. \tag{39}$$

Since the diagonal representation is known to be unique,[3] the functional defined by Eq. (39) is equivalent, in the sense of generalized functions and in the space Z', to the well-known Gaussian representation for the thermal-field weight function[5]

$$\varphi(r) = \frac{e^{-r^2/\langle n \rangle}}{\pi \langle n \rangle}; \quad \langle n \rangle = \frac{e^{-\beta}}{(1 - e^{-\beta})}, \tag{40}$$

thus confirming a conjecture by Kano.[20]

III. DISCUSSION

For stationary fields, we have shown that the infinite series Eq. (22) is absolutely convergent, and hence the expansion Eq. (8) defines a functional on the space $Z(R_2)$. Using the general expansion Eq. (6), we obtain the following result for an arbitrary density operator analogous to Eq. (22):

$$\begin{aligned}\langle \varphi(r, \theta), \psi(r, \theta) \rangle &= \sum_{n,m=0}^{\infty} \langle n | \hat{\rho} | m \rangle e^{i(n-m)(\theta+\pi)} \times \sum_{k=0}^{(n+m/2)} \frac{(n!\, m!)^{\frac{1}{2}}}{\left[\frac{(n+m)}{2} - k\right]!\,(2k)!} \psi^{(2k)}(0, \theta) \qquad (41) \\ &= \sum_{n,m=0}^{\infty} \langle n | \hat{\rho} | m \rangle e^{i(n-m)(\theta+\pi)} \frac{(n!\, m!)^{\frac{1}{2}}}{\left(\frac{n+m}{2}\right)!} \times \iint M\left(-\frac{(n+m)}{2}, \frac{1}{2}, \left\{\frac{\rho \cos(\theta - \gamma)}{2}\right\}^2\right) \times \tilde{\psi}(\rho, \gamma) \rho \, d\rho \, d\gamma. \qquad (42)\end{aligned}$$

[16] A. H. Zemanian, *Distribution Theory and Transform Calculus* (McGraw-Hill Book Co., Inc., New York, 1965), p. 195.

[17] Reference 13, p. 780, formula (22.5.56).

[18] Reference 13, p. 787, formula (22.14.17).

[19] A. Messiah, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961), Vol. I, p. 337.

[20] Y. Kano, Proc. Phys. Soc. (Japan) **19**, 1555 (1964).

Using Stirling's approximation for $n!$

$$n! \sim (2\pi n)^{\frac{1}{2}} \left(\frac{n}{e}\right)^n, \quad n \to \infty, \tag{43}$$

it follows that, for $m = 0$, $n \to \infty$, the factor

$$\frac{(n!\, m!)^{\frac{1}{2}}}{\left(\frac{n+m}{2}\right)!} = \frac{(n!)^{\frac{1}{2}}}{\left(\frac{n}{2}\right)!} \sim \frac{\left(\frac{n}{e}\right)^{n/2}}{\left(\frac{n}{2e}\right)^{n/2}} = 2^{n/2},$$

and therefore the series (42), is not absolutely convergent for arbitrary density operators. In order to obtain a result analogous to (22), we must exploit the conditional convergence of this series. This result, however, has not yet been established.

One final remark is in order. Although the original proof of the existence of a diagonal representation for arbitrary density operators was based on expansion (6), an alternate technique for defining the weight functional has subsequently been introduced.[21–24] In this approach,[23,24] the weight $\varphi(z)$, which is, in general, a *functional* in the space $Z'(R_2)$, is defined as the limit of an infinite sequence of infinitely differentiable *functions* of rapid decrease. From a conceptual point of view, it is no doubt true that such well-behaved functions are easier to assimilate than functionals in Z', and hence the latter approach to the problem of the diagonal representation is preferable in this sense to the original formulation. However, we stress that the results of Sec. II demonstrate that the definition of $\varphi(z)$ based on the infinite series of the delta function and its derivatives is just as rigorous from a mathematical point of view as the sequential approach, at least for stationary fields.

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The author is grateful to Professor J. R. Klauder for stimulating discussions and suggestions.

[21] J. R. Klauder, J. McKenna, and D. G. Currie, J. Math. Phys. **6**, 733 (1965).
[22] C. L. Mehta and E. C. G. Sudarshan, Phys. Rev. **138**, B274 (1965).
[23] J. R. Klauder, Phys. Rev. Letters **16**, 534 (1966).
[24] F. Rocca, Compt. Rend. **262**, A547 (1966).

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"Lorentz Basis" of the Poincaré Group

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An explicit derivation is given for the matrix elements of the translation generators P_μ of the Poincaré algebra with respect to the "Lorentz basis," namely, in terms of states which diagonalize the two Casimir operators of the homogeneous Lorentz group (HLG). The results are given for the cases mass $\mu > 0$ and $\mu = 0$ and, for the latter, for discrete and continuous spin. The transforms connecting the momentum and Lorentz bases are discussed, a detailed derivation being given for the zero-mass discrete-spin case. The matrix elements of $G_\mu = i[(\mathbf{N}^2 - \mathbf{M}^2), P_\mu]$ are considered and several interesting aspects of the algebras generated by $\mathbf{N}$, $\mathbf{M}'$, and $P'_\mu = (\epsilon_1 P_\mu + \epsilon_2 G_\mu)$ are discussed for the cases of positive as well as zero rest mass.

1. INTRODUCTION

In the following section we give an explicit construction of the matrix elements of the translation operators of the Poincaré algebra with respect to the "Lorentz basis." By Lorentz basis we mean that which diagonalizes the two Casimir operators of the homogeneous Lorentz group.[1,2] This derivation is quite formal in the sense that the "pure states" of the unitary representation we start with do not give a "basis" in the conventional sense. This is most clearly displayed by the fact that the action of the generators P_μ on a pure state belonging to the eigenvalue λ (real for unitary representation) of the operator $(\mathbf{N}^2 - \mathbf{M}^2)$ gives us formally the states corresponding to $(\lambda \pm i)$. These matrix elements are not Hermitic. A coherent formalism may, for example, be achieved in terms of suitably "smeared" basis vectors. The above feature is a consequence of the noncompactness of the diagonalized operators of the homogeneous part in semidirect product with the translation subgroup.[3] It does not arise when only $\mathbf{M}^2$ (corresponding to the compact

[1] H. Joos, Fortschr. Phys. **10**, 65 (1962) (particularly pp. 88–109).
[2] I. M. Gel'fand, R. A. Milnos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (Pergamon Press, Inc., New York, 1963).
[3] E. C. G. Sudarshan, "Noncompact Groups in Particle Physics" *Coral Gables Conference 1966* (W. H. Freeman and Co., San Francisco, 1966), p. 254.

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it follows that, for $m = 0$, $n \to \infty$, the factor

$$\frac{(n!\,m!)^{\frac{1}{2}}}{\left(\frac{n+m}{2}\right)!} = \frac{(n!)^{\frac{1}{2}}}{\left(\frac{n}{2}\right)!} \sim \frac{\left(\frac{n}{e}\right)^{n/2}}{\left(\frac{n}{2e}\right)^{n/2}} = 2^{n/2},$$

and therefore the series (42), is not absolutely convergent for arbitrary density operators. In order to obtain a result analogous to (22), we must exploit the conditional convergence of this series. This result, however, has not yet been established.

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An explicit derivation is given for the matrix elements of the translation generators P_μ of the Poincaré algebra with respect to the "Lorentz basis," namely, in terms of states which diagonalize the two Casimir operators of the homogeneous Lorentz group (HLG). The results are given for the cases mass $\mu > 0$ and $\mu = 0$ and, for the latter, for discrete and continuous spin. The transforms connecting the momentum and Lorentz bases are discussed, a detailed derivation being given for the zero-mass discrete-spin case. The matrix elements of $G_\mu = i[(\mathbf{N}^2 - \mathbf{M}^2), P_\mu]$ are considered and several interesting aspects of the algebras generated by $\mathbf{N}$, $\mathbf{M}'$, and $P'_\mu = (\epsilon_1 P_\mu + \epsilon_2 G_\mu)$ are discussed for the cases of positive as well as zero rest mass.

1. INTRODUCTION

In the following section we give an explicit construction of the matrix elements of the translation operators of the Poincaré algebra with respect to the "Lorentz basis." By Lorentz basis we mean that which diagonalizes the two Casimir operators of the homogeneous Lorentz group.[1,2] This derivation is quite formal in the sense that the "pure states" of the unitary representation we start with do not give a "basis" in the conventional sense. This is most clearly displayed by the fact that the action of the generators P_μ on a pure state belonging to the eigenvalue λ (real for unitary representation) of the operator $(\mathbf{N}^2 - \mathbf{M}^2)$ gives us formally the states corresponding to $(\lambda \pm i)$. These matrix elements are not Hermitic. A coherent formalism may, for example, be achieved in terms of suitably "smeared" basis vectors. The above feature is a consequence of the noncompactness of the diagonalized operators of the homogeneous part in semidirect product with the translation subgroup.[3] It does not arise when only $\mathbf{M}^2$ (corresponding to the compact

[1] H. Joos, Fortschr. Phys. **10**, 65 (1962) (particularly pp. 88–109).
[2] I. M. Gel'fand, R. A. Milnos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and Their Applications* (Pergamon Press, Inc., New York, 1963).
[3] E. C. G. Sudarshan, "Noncompact Groups in Particle Physics" *Coral Gables Conference 1966* (W. H. Freeman and Co., San Francisco, 1966), p. 254.

rotation subgroup) is diagonalized, as in the "helicity"[4] or the "L-S"[5] basis of the Poincaré algebra. There we obtain, at least formally, Hermitic matrix elements.

Even in the momentum basis (with its continuous and unbounded spectrum), a consistent mathematical interpretation necessitates a generalization of the usual Hilbert space formalism (as, for example, that proposed in Ref. 6). As noted above, additional features arise in our case.

Nevertheless, as a starting point, it is quite useful to have the explicit forms of the matrix elements and we give the results for the cases of positive and zero mass,[7] and for both discrete and continuous spin for the latter case.

In Sec. 3, we study the transformations between the momentum and Lorentz basis and demonstrate one possible utility of the matrix elements by taking the zero-mass case as an example. These formulas, of course, incidentally furnish one way of obtaining formally Hermitic matrix elements of the P_μ, through a linear superposition of the pure "Lorentz states"—namely, by constructing in their terms the momentum eigenstates themselves. Also it is seen that the definition of the action of P_μ on the Lorentz states, through an integral over the momentum states, formally involves the matrix elements of Sec. 2.

Another application of the matrix elements is illustrated in Sec. 4. In fact, in the study of deformations of the Poincaré algebra[8–10] which leave intact the homogeneous part, the Lorentz basis has a "natural" place. Apart from such studies, several attempts have recently been made,[11,12] from different points of view, to use the unitary representations of the HLG in the theory of particles. It is our impression that the formal aspects studied in this paper should be of interest for such attempts.

A discussion of the results obtained is given in Sec. 5.

2. MATRIX ELEMENTS

In this section we propose to determine explicitly the matrix elements of the translation operators P_μ with respect to the canonical basis[1,2] of the homogeneous Lorentz group (HLG). The matrix elements of the generators of the HLG are given by

$$M_3\,|jm\rangle_{j_0\lambda} = m\,|jm\rangle_{j_0\lambda}, \quad (M_1 \pm iM_2)\,|jm\rangle_{j_0\lambda} = [(j \mp m)(j \pm m + 1)]^{\frac{1}{2}}\,|jm \pm 1\rangle_{j_0\lambda},$$

$$\begin{aligned} N_3\,|jm\rangle_{j_0\lambda} = {} & \frac{1}{j+1}\left\{\frac{[(j+1)^2 - j_0^2][(j+1)^2 + \lambda^2][(j+1)^2 - m^2]}{(2j+3)(2j+1)}\right\}^{\frac{1}{2}}|j+1\,m\rangle_{j_0\lambda} \\ & + \frac{j_0\lambda}{j(j+1)}\,m\,|jm\rangle_{j_0\lambda} + \frac{1}{j}\left[\frac{(j^2 - j_0^2)(j^2 + \lambda^2)(j^2 - m^2)}{(2j+1)(2j-1)}\right]^{\frac{1}{2}}|j-1\,m\rangle_{j_0\lambda}, \end{aligned} \tag{2.1}$$

$$\begin{aligned} (N_1 \pm iN_2)\,|jm\rangle_{j_0\lambda} = {} & \mp\frac{1}{j+1}\left\{\frac{[(j+1)^2 - j_0^2][(j+1)^2 + \lambda^2](j \pm m + 2)(j \pm m + 1)}{(2j+3)(2j+1)}\right\}^{\frac{1}{2}}|j+1, m \pm 1\rangle_{j_0\lambda} \\ & + j_0\lambda\left\{\frac{(j \pm m + 1)(j \mp m)}{[j(j+1)]^2}\right\}^{\frac{1}{2}}|j, m \pm 1\rangle_{j_0\lambda} \\ & \pm\frac{1}{j}\left[\frac{(j^2 - j_0^2)(j^2 + \lambda^2)(j \mp m - 1)(j \mp m)}{(2j+1)(2j-1)}\right]^{\frac{1}{2}}|j-1, m \pm 1\rangle_{j_0\lambda}. \end{aligned} \tag{2.2}$$

The two Casimir operators for the homogeneous part are given by

$$\begin{aligned} (\mathbf{N}^2 - \mathbf{M}^2)\,|jm\rangle_{j_0\lambda} &= (1 + \lambda^2 - j_0^2)\,|jm\rangle_{j_0\lambda}, \\ \mathbf{N}\cdot\mathbf{M}\,|jm\rangle_{j_0\lambda} &= j_0\lambda\,|jm\rangle_{j_0\lambda}. \end{aligned} \tag{2.3}$$

Noting that the substitution

$$(j_0, \lambda) \to (-j_0, -\lambda) \tag{2.4}$$

leaves the above matrix elements unaffected, we can classify the nontrivial unitary representations of HLG as

$$\begin{aligned} &(1) \quad j_0 = 0, \quad \lambda \geq 0, \\ &(2) \quad j_0 = 1, 2, 3, \cdots; \quad -\infty < \lambda < +\infty, \\ &(3) \quad j_0 = \tfrac{1}{2}, \tfrac{3}{2}, \cdots; \quad -\infty < \lambda < +\infty \quad \text{(double-valued)}, \\ &(4) \quad j_0 = 0, \quad 0 < -i\lambda < 1. \end{aligned} \tag{2.5}$$

[4] J. S. Lomont and H. E. Moses, J. Math. Phys. **5**, 294 and 1438 (1964).
[5] A. Chakrabarti, J. Math. Phys. **7**, 426 (1966).
[6] I. M. Gel'fand and N. Ya. Vilenkin, *Generalized Functions* (Academic Press, Inc., New York, 1964), Vol. 4, Chap. I, Sec. 4.
[7] H. Joos, Revista Univ. Mat. Arg. **18**, 119 (1956). While Ref. 1 treats only the case of positive rest mass, in Ref. 7 both cases, namely, positive and zero rest mass, have been treated, but only for spin zero and using a different basis.
[8] A. Böhm, "The Dynamical Group of a Simple Particle Model," Syracuse Univ. Preprint (NYO-3399-77-1206-SU-77).
[9] R. Hermann, Commun. Math. Phys. **3**, 53 (1966).
[10] M. Lévy-Nahas, J. Math. Phys. **8**, 1211 (1967).
[11] For the use of wave equations corresponding to the unitary representations of HLG way refer, for example, to Y. Nambu, Suppl. Progr. Theoret. Phys. (Kyoto) Nos. 37 and 38 (1966), p. 368; A. Barut and H. Kleinert, Preprint Univ. Col. 66-13, Boulder, Colorado, and the sources quoted therein.
[12] For the use of analyticity properties associated with reduction of the scattering amplitudes w.r.t. the HLG see, M. Toller, Nuovo Cimento **54A**, 295 (1968); R. Delbourgo, A. Salam, and J. Strathdee, Phys. Rev. **164**, 1981 (1967) and the references quoted in these papers.

The above matrix elements correspond to the normalization

$$ {}_{j_0'\lambda'}\langle j'm' \mid jm\rangle_{j_0\lambda} = \delta_{jj'}\delta_{mm'}\delta_{j_0j_0'}\delta(\lambda - \lambda'). \quad (2.5') $$

With the notations thus fixed, let us start by considering the operator P_0. Noting that P_0 commutes with M^2 and M_3, let us write

$$ P_0 \mid jm\rangle_\tau = \sum_{\tau'} c_j^{\tau'\tau} \mid jm\rangle_{\tau'} \quad [\tau : (j_0, \lambda)]. \quad (2.6) $$

Utilizing

$$ [N_3, [N_3, P_0]] = -P_0, \quad (2.7) $$

we find that the only nonzero matrix elements correspond to

$$ \tau' = (j_0 \pm 1, \lambda) \quad [j_0, (\lambda \pm i)], \quad (2.8) $$

and for these cases

$$ c_j^{\tau'\tau} = c^{\tau'\tau}[(j \mp j_0)(j \pm j_0 + 1)]^{\frac{1}{2}} \quad \text{for} \quad j_0' = j_0 \pm 1, \quad \lambda' = \lambda, \quad (2.9) $$

and

$$ c_j^{\tau'\tau} = c^{\tau'\tau}[(j \pm i\lambda)(j \mp i\lambda + 1)]^{\frac{1}{2}} \quad \text{for} \quad j_0' = j_0, \quad \lambda' = \lambda \pm i. \quad (2.10) $$

(In fact this part of the calculation is formally the same as that presented in pp. 275–6 of Ref. 2. In our case however we cannot leave $c^{\tau'\tau}$ arbitrary.)

In order to determine the remaining factors, it is convenient to use the relations

$$ [P_0, [P_0, (\mathbf{N}^2 - \mathbf{M}^2)]] = -2(P_0^2 - \mu^2), $$
$$ [P_0, [P_0, \mathbf{N}\cdot\mathbf{M}]] = 0, \quad (2.11) $$

where μ denotes the mass corresponding to the irreducible representation of the Poincaré group and may be positive or zero for the cases to be considered.

From Eq. (2.11) we obtain the equations

$$ \sum_{\tau'} c_j^{\tau''\tau'} c_j^{\tau'\tau}(F_\tau + F_{\tau''} - 2F_{\tau'} + 2) = 2\mu^2\delta_{\tau\tau''}, $$
$$ \sum_{\tau'} c_j^{\tau''\tau'} c_j^{\tau'\tau}(G_\tau + G_{\tau''} - 2G_{\tau'}) = 0, \quad (2.12) $$

where

$$ F_\tau \equiv (1 + \lambda^2 - j_0^2), \quad G_\tau \equiv j_0\lambda. \quad (2.13) $$

Case: $\mu > 0$

In this case, putting $\tau'' = \tau$ in Eq. (2.12) and putting

$$ \alpha_\lambda(j_0, \lambda) \equiv c^{(j_0,\lambda),(j_0,\lambda+i)} c^{(j_0,\lambda+i),(j_0,\lambda)}, $$
$$ \beta_{j_0}(j_0, \lambda) \equiv c^{(j_0\lambda),(j_0+1,\lambda)} c^{(j_0+1,\lambda),(j_0,\lambda)}, \quad (2.14) $$

we obtain the following set of relations:

$$ [j_0(\alpha_\lambda - \alpha_{\lambda-i}) - i\lambda(\beta_{j_0} - \beta_{j_0-1})] = 0, $$
$$ [-i\lambda(\alpha_\lambda - \alpha_{\lambda-i}) + j_0(\beta_{j_0} - \beta_{j_0-1}) + (\alpha_\lambda + \alpha_{\lambda-i}) + (\beta_{j_0} + \beta_{j_0-1})] = 0, \quad (2.15) $$
$$ \{i\lambda[\alpha_\lambda(-i\lambda + 1)^2 - \alpha_{\lambda-i}(i\lambda + 1)^2] - j_0[\beta_{j_0}(j_0 + 1)^2 - \beta_{j_0-1}(j_0 - 1)^2]\} = \tfrac{1}{2}\mu^2. $$

The solutions are

$$ \alpha_\lambda = \frac{\zeta^2 - \mu^2(1 - 2i\lambda)^2}{16(j_0 + i\lambda)(j_0 - i\lambda + 1)(j_0 - i\lambda)(j_0 + i\lambda - 1)}, \quad (2.16) $$

$$ \beta_{j_0} = \frac{\zeta^2 - \mu^2(1 + 2j_0)^2}{16(j_0 + i\lambda)(j_0 - i\lambda + 1)(j_0 - i\lambda)(j_0 + i\lambda + 1)}. \quad (2.17) $$

The parameter ζ, as yet undetermined, is to be evaluated in terms of the spin corresponding to the irreducible representations [see Eqs. (2.24)–(2.26)].

The reduced elements corresponding to the transitions $\lambda \to \lambda - i$ and $j_0 \to j_0 - 1$ are given by $\alpha_{\lambda-i}$ and β_{j_0-1}.

Writing (with a slightly altered and obvious notation for the reduced elements)

$$ e^{-i\varphi} c_{j_0}^{\lambda+i,\lambda} = e^{i\varphi} c^{\lambda,\lambda+i} = (\alpha_\lambda)^{\frac{1}{2}}, $$
$$ e^{-i\varphi'} c_\lambda^{j_0+1,j_0} = e^{i\varphi'} c_\lambda^{j_0,j_0+1} = (\beta_{j_0})^{\frac{1}{2}} \quad (2.18) $$

(where the phases φ, φ' may be chosen according to some suitable convention), all the relations in Eq. (2.12), for different possible values of τ'', may be seen to be satisfied automatically. In terms of Eq. (2.16)–(2.18), we have

$$ \begin{aligned} P_0 \mid jm\rangle_{j_0\lambda} = {} & c_\lambda^{j_0+1,j_0}[(j - j_0)(j + j_0 + 1)]^{\frac{1}{2}} \mid jm\rangle_{j_0+1,\lambda} \\ & + c_\lambda^{j_0-1,j_0}[(j + j_0)(j - j_0 + 1)]^{\frac{1}{2}} \mid jm\rangle_{j_0-1,\lambda} \\ & + c_{j_0}^{\lambda+i,\lambda}[(j + i\lambda)(j - i\lambda + 1)]^{\frac{1}{2}} \mid jm\rangle_{j_0,\lambda+i} \\ & + c_{j_0}^{\lambda-i,\lambda}[(j - i\lambda)(j + i\lambda + 1)]^{\frac{1}{2}} \mid jm\rangle_{j_0,\lambda-i}. \end{aligned} \quad (2.19) $$

The matrix elements of $\mathbf{P}$ now may be written down directly by using the relation

$$ \mathbf{P} = i[\mathbf{N}, P^0]. \quad (2.20) $$

For example, we obtain

$$\begin{aligned}
P_3 |jm\rangle_{j_0\lambda} = \sum_{\pm} \Bigg\{ & -\frac{i}{j+1}\left[\frac{(j+1)^2 - m^2}{(2j+3)(2j+1)}\right]^{\frac{1}{2}} \\
& \times \{[((j+1)^2 + \lambda^2)(j \pm j_0 + 1)(j \pm j_0 + 2)]^{\frac{1}{2}} c_\lambda^{j_0+1,j_0} | j + 1 m\rangle_{j_0 \pm i\lambda} \\
& + [((j+1)^2 - j_0^2)(j \mp i\lambda + 1)(j \mp i\lambda + 2)]^{\frac{1}{2}} c_{j_0}^{\lambda \pm i,\lambda} | j + 1 m\rangle_{j_0,\lambda \pm i}\} \\
& \pm \frac{m}{j(j+1)} \{i\lambda[(j \mp j_0)(j \pm j_0 + 1)]^{\frac{1}{2}} c_\lambda^{j_0 \pm 1, j_0} |jm\rangle_{j_0 \pm 1,\lambda} \\
& - j_0[(j \pm i\lambda)(j \mp i\lambda + 1)]^{\frac{1}{2}} c_{j_0}^{\lambda \pm i,\lambda} |jm\rangle_{j_0,\lambda \pm i}\} \\
& + \frac{i}{j}\left[\frac{j^2 - m^2}{(2j+1)(2j-1)}\right]^{\frac{1}{2}} \{[(j^2 + \lambda^2)(j \mp j_0)(j \mp j_0 - 1)]^{\frac{1}{2}} c_\lambda^{j_0 \pm 1, j_0} | j - 1 m\rangle_{j_0 \pm 1,\lambda} \\
& + [(j^2 - j_0^2)(j \pm i\lambda)(j \pm i\lambda - 1)]^{\frac{1}{2}} c_{j_0}^{\lambda \pm i,\lambda} | j - 1, m\rangle_{j_0,\lambda \pm i}\}\Bigg\}. \qquad (2.21)
\end{aligned}$$

In order to introduce the spin parameter explicitly in the matrix elements, we have to evaluate

$$W_\mu = -i[P_\mu, \mathbf{N} \cdot \mathbf{M}]. \qquad (2.22)$$

Thus, for example,

$$\begin{aligned}
W_0 |jm\rangle_{j_0\lambda} &= i\lambda\{c_\lambda^{j_0+1,j_0}[(j - j_0)(j + j_0 + 1)]^{\frac{1}{2}} |jm\rangle_{j_0+1,\lambda} \\
&\quad - c_\lambda^{j_0-1,j_0}[(j + j_0)(j - j_0 + 1)]^{\frac{1}{2}} |jm\rangle_{j_0-1,\lambda}\} \\
&\quad - j\{c_{j_0}^{\lambda+i,\lambda}[(j + i\lambda)(j - i\lambda + 1)]^{\frac{1}{2}} |jm\rangle_{j_0\lambda+i} \\
&\quad - c_{j_0}^{\lambda-i,\lambda}[(j - i\lambda)(j + i\lambda + 1)]^{\frac{1}{2}} |jm\rangle_{j_0\lambda-i}\}. \qquad (2.23)
\end{aligned}$$

Finally we obtain

$$W^2 |jm\rangle_{j_0\lambda} = -\tfrac{1}{4}(\zeta^2 - \mu^2) |jm\rangle_{j_0\lambda}. \qquad (2.24)$$

Hence, for

$$W^2 = -\mu^2 s(s+1)I,$$
$$\zeta^2 = \mu^2(4s(s+1) + 1), \qquad (2.25)$$

and consequently

$$\alpha_\lambda = \frac{\mu^2}{4}\frac{(s + i\lambda)(s - i\lambda + 1)}{(j_0 + i\lambda)(j_0 - i\lambda + 1)(j_0 - i\lambda)(j_0 + i\lambda - 1)},$$
$$\beta_{j_0} = \frac{\mu^2}{4}\frac{(s - j_0)(s + j_0 + 1)}{(j_0 + i\lambda)(j_0 - i\lambda + 1)(j_0 - i\lambda)(j_0 + i\lambda + 1)}. \qquad (2.26)$$

Thus we see that if we start, corresponding to integral or half-integral value of s, with an integral or half-integral value, respectively, of j_0, such that

$$-s \leq j_0 \leq s, \qquad (2.27)$$

then we have nonzero matrix elements only for

$$j_0 = -s, -s + 1, \cdots, s. \qquad (2.28)$$

In particular, for

$$s = 0,$$

we can take

$$j_0 = 0$$

and be left with only the matrix elements

$$\lambda \to \lambda \pm i,$$

with

$$\alpha_\lambda = \frac{\mu^2}{4}\frac{1}{(-i\lambda)(i\lambda - 1)} = \frac{\mu^2}{4}\frac{1}{\lambda(\lambda + i)}. \qquad (2.29)$$

However, as formal solutions of the recurrence relations, we also obtain the following possible domains of variation

$$s \leq j_0 < \infty \quad \text{and} \quad -\infty < j_0 \leq -s. \qquad (2.30)$$

Only the case (2.28) will be utilized for the integral transforms considered in Sec. 3, where some relevant comments are added.

Case: $\mu = 0$

(i) Discrete Spin

As may be verified directly and easily, we have four possible solutions in this case:

$$P_0 |jm\rangle_{j_0\lambda} = a_1[(j - i\lambda)(j + i\lambda + 1)]^{\frac{1}{2}} |jm\rangle_{j_0,\lambda-i}, \qquad (2.31a)$$

$$P_0 |jm\rangle_{j_0\lambda} = a_2[(j + i\lambda)(j - i\lambda + 1)]^{\frac{1}{2}} |jm\rangle_{j_0,\lambda+i}, \qquad (2.31b)$$

$$P_0 |jm\rangle_{j_0\lambda} = b_1[(j + j_0)(j - j_0 + 1)]^{\frac{1}{2}} |jm\rangle_{j_0-1,\lambda}, \qquad (2.31c)$$

$$P_0 |jm\rangle_{j_0\lambda} = b_2[(j - j_0)(j + j_0 + 1)]^{\frac{1}{2}} |jm\rangle_{j_0+1,\lambda}. \qquad (2.31d)$$

The parameters $a_{1,2}$ and $b_{1,2}$ are as yet arbitrary (independent, of course, of j and m).

If we choose the $a_{1,2}$ to be functions of j_0 only and the $b_{1,2}$ to be functions of λ only, they become constants. In fact, in the above cases, with only one state on the right-hand side, it is easy to give the matrix elements of P_0 its simplest form. For example, putting $a_1 = 1$ in Eq. (2.31) and defining

$$|jm\rangle'_{j_0\lambda} = \left(\frac{(j+i\lambda)!}{(j-i\lambda)!}\right)^{\frac{1}{2}} |jm\rangle_{j_0\lambda}, \qquad (2.32)$$

with a corresponding change in Eq. (2.32), we obtain

$$P_0 \, |jm\rangle'_{j_0\lambda} = |jm\rangle_{j_0\lambda - i}. \qquad (2.33)$$

For the solutions (2.31a)–(2.31d) we have, from Eq. (2.22),

$$W_\mu = \pm j_0 P_\mu, \quad \pm(i\lambda)P_\mu, \qquad (2.34)$$

respectively.

Thus we see that, corresponding to unitary representations of HLG, Eq. (2.5), only the first two give physical values ($\pm j_0$) for the *invariant* helicity. Moreover, for the same value of j_0, the passage from one to the other corresponds to a change in sign of the helicity.

(*ii*) *Continuous Spin*

If in Eqs. (2.16) and (2.17) we put

$$\mu = 0 \quad \text{and} \quad \zeta > 0, \qquad (2.35)$$

we obtain the solutions for the case

$$P^2 = 0, \quad W^2 = -\tfrac{1}{4}\zeta^2 I; \qquad (2.36)$$

namely, that of zero mass and continuous spin.

3. THE TRANSFORMS CONNECTING THE MOMENTUM AND THE LORENTZ BASIS

The required formulas have been given by Joos[1] for the case $\mu > 0$ for the canonical Lorentz basis we are considering. (For the corresponding transform using a different basis, see Ref. 13 and the references quoted therein.)

(*i*) $\mu = 0$ (*Discrete Spin*)

However, if one has at hand the explicit matrix elements of P^μ to start with (Sec. 2), they can be usefully employed to extract all the factors which do not depend *only* upon the p_ν's (in fact, only upon p_0, since the rotation dependence is easily made explicit). In order to illustrate this point, let us consider the relatively simple case $\mu = 0$, choosing in particular the subcase $a_1 = 1$ for the sake of simplicity

$$P_0 \, |jm\rangle_{j_0\lambda} = [(j - i\lambda)(j + i\lambda + 1)]^{\frac{1}{2}} \, |jm\rangle_{j_0\lambda - i}, \quad (3.1)$$

where

$$W_\mu = j_0 P_\mu \qquad (3.2)$$

gives the helicity.

Let an eigenket in the momentum basis be denoted by $|\mathbf{p}\rangle_{j_0}$ and let us assume

$$|\mathbf{p}\rangle_{j_0} = \sum_{jm} \int_{-\infty}^{\infty} d\lambda \left[\mathfrak{D}^j_{mj_0}(\varphi, \theta, -\varphi) \left(\frac{2j+1}{4\pi}\right)^{\frac{1}{2}} \right] F_{j\lambda}(p) \, |jm\rangle_{j_0\lambda} \quad (p = |\mathbf{p}| = p_0). \quad (3.3)$$

This ansatz evidently ensures the correct matrix elements of $\mathbf{M}$, and we have introduced the factor $[(2j+1)/4\pi]^{\frac{1}{2}}$ for the sake of convenience. It remains for us to determine the form of $F_{j\lambda}(p)$.

Now applying P_0 to both sides (which just multiplies the left-hand side by p) and using Eq. (3.1), we obtain at once

$$F_{j\lambda}(p) = p^{-i\lambda} \left(\frac{(j+i\lambda)!}{(j-i\lambda)!}\right)^{\frac{1}{2}} F_j(p). \qquad (3.4)$$

Now applying $P^3 = i[N^3, P_0]$ to both sides of Eq. (3.3) and using the relation

$$\begin{aligned} &\cos\theta[(2j+1)^{\frac{1}{2}}\mathfrak{D}^j_{mj_0}] \\ &= \frac{1}{j+1}\left\{\frac{[(j+1)^2 - m^2][(j+1)^2 - j_0^2]}{(2j+3)(2j+1)}\right\}^{\frac{1}{2}} \\ &\quad \times [(2j+3)^{\frac{1}{2}}\mathfrak{D}^{j+1}_{mj_0}] + \frac{mj_0}{j(j+1)}[(2j+1)^{\frac{1}{2}}\mathfrak{D}^j_{mj_0}] \\ &\quad + \frac{1}{j}\left[\frac{(j^2 - m^2)(j^2 - j_0^2)}{(2j+1)(2j-1)}\right]^{\frac{1}{2}}[(2j-1)^{\frac{1}{2}}\mathfrak{D}^{j-1}_{mj_0}], \end{aligned} \quad (3.5)$$

we obtain

$$F_j(p) = (i)^{-j} F(p). \qquad (3.6)$$

The simple recurrence relations implied by the matrix elements of P^μ can lead us only this far. In order to evaluate $F(p)$, we have to consider the action of some operator not commuting with P^0, namely, $\mathbf{N}$. In this particular case, however, our task is fairly simple.

Using the canonical generators in the momentum basis[14] for $\mu = 0$ [with a change in sign of $\mathbf{N}$ in order to conform to the convention of Joos[1] where $\mathbf{N}$ consists of the components (m_{0i}, $i = 1, 2, 3$) instead of (M^{0i}) as in Ref. 14]

$$\mathbf{N} = \left(iP^0 \frac{\partial}{\partial \mathbf{P}} + \frac{\mathbf{P} \times \mathbf{S}}{P^0}\right),$$

$$\mathbf{M} = -i\mathbf{P} \times \frac{\partial}{\partial \mathbf{P}} + \mathbf{S}, \qquad (3.7)$$

we obtain at once

$$\mathbf{N} \cdot \mathbf{M} = i\left(\mathbf{P} \cdot \frac{\partial}{\partial \mathbf{P}} + 1\right)\left(\frac{\mathbf{S} \cdot \mathbf{P}}{|\mathbf{P}|}\right) \qquad (3.8)$$

and

$$\mathbf{N}^2 - \mathbf{M}^2 = (\mathbf{N}_0^2 - \mathbf{M}_0^2) - \left(\frac{\mathbf{S} \cdot \mathbf{P}}{|\mathbf{P}|}\right)^2, \qquad (3.9)$$

[13] Chou Kuang-Chao and L. G. Zastavenko, Sov. Phys.—JETP **8**, 990 (1956); **35**, 1417 (1958); V. S. Popov, Sov. Phys.—JETP **10**, 794 (1960); **37**, 1116 (1959).

[14] A. Chakrabarti, J. Math. Phys. **7**, 949, 1966.

where the orbital part

$$(\mathbf{N}_0^2 - \mathbf{M}_0^2) = -\left(p^2 \frac{d^2}{dp^2} + 3p\frac{d}{dp}\right).$$
$$= 1 + \left[i\left(\mathbf{p}\cdot\frac{\partial}{\partial \mathbf{p}} + 1\right)\right]^2. \quad (3.10)$$

The operators corresponding to j_0 and λ can now be read off from (3.8) and (3.10).

Using the equivalent explicitly one-dimensional form the representation,[14,15] we, of course, obtain essentially the same expression, with the helicity parameter (in our case j_0) replacing $\mathbf{S}\cdot\mathbf{P}/|\mathbf{P}|$.

Using either Eqs. (3.8) or (3.9), we obtain, in a consistent fashion [instead of the Legendre functions of the $\mu > 0$ case arising from Eq. (4.13) of Ref. 1],

$$F(p) = Cp^{-1} \quad \text{or} \quad F_{j\lambda}(p) = C\left[\frac{(j+i\lambda)!}{(j-i\lambda)!}\right]^{\frac{1}{2}}(i)^{-j}p^{-i\lambda-1}, \quad (3.11)$$

where C is a normalizing constant to be determined.

In fact, in order to extract the j, λ dependence of F, we could have also used N^3 instead of P^3. But then, instead of Eq. (3.5) we would have needed a somewhat different relation, namely,

$$\sin\theta\frac{d}{d\theta}\mathfrak{D}^j_{mj_0} = \frac{j}{(j+1)(2j+1)} \times \{[(j+1)^2 - m^2][(j+1)^2 - j_0^2]\}^{\frac{1}{2}}\mathfrak{D}^{j+1}_{mj_0} - \frac{mj_0}{j(j+1)}\mathfrak{D}^j_{mj_0} - \frac{j+1}{j(2j+1)}[(j^2-m^2)(j^2-j_0^2)]^{\frac{1}{2}}\mathfrak{D}^{j-1}_{mj_0}. \quad (3.12)$$

Thus, we have

$$|\mathbf{p}\rangle_{j_0} = C\sum_{jm}\int_{-\infty}^{\infty} d\lambda (i)^{-j}\left(\frac{(j+i\lambda)!}{(j-i\lambda)!}\right)^{\frac{1}{2}} \times \left(\frac{2j+1}{4\pi}\right)^{\frac{1}{2}}\mathfrak{D}^j_{mj_0}(\varphi,\theta,-\varphi)p^{-i\lambda-1}\,|jm\rangle_{j_0\lambda}. \quad (3.13)$$

C is fixed in order to insure

$${}_{j_0'}\langle\mathbf{p}' \mid \mathbf{p}\rangle_{j_0} = \delta_{j_0'j_0}2p\delta(\mathbf{p}-\mathbf{p}')$$

and

$${}_{j_0'\lambda'}\langle j'm' \mid jm\rangle_{j_0\lambda} = \delta_{j'j}\delta_{m'm}\delta_{j_0'j_0}\delta(\lambda'-\lambda). \quad (3.14)$$

Noting that

$$\frac{1}{pp'}\int_{-\infty}^{\infty} d\lambda\left(\frac{p'}{p}\right)^{i\lambda} = \frac{2\pi}{p}\delta(p-p'), \quad (3.15)$$

$$\sum_{jm}\frac{2j+1}{4\pi}\mathfrak{D}^{j\dagger}_{mj_0}(\Omega')\mathfrak{D}^j_{mj_0}(\Omega) = \delta(\Omega-\Omega')$$
$$= \sin\theta\delta(\theta-\theta')\delta(\varphi-\varphi') \quad [\Omega\to(\varphi,\theta,-\varphi)], \quad (3.16)$$

and

$$\delta(p-p')\delta(\Omega-\Omega') = p^2\delta(\mathbf{p}-\mathbf{p}'), \quad (3.17)$$

we obtain

$$C^2 = 1/\pi. \quad (3.18)$$

Thus we may write

$$|\mathbf{p}\rangle_{j_0} = \frac{1}{2\pi}\sum_{jm}\int d\lambda(i)^{-j}\left[\frac{(j+i\lambda)!}{(j-i\lambda)!}\right]^{\frac{1}{2}} \times (2j+1)^{\frac{1}{2}}\mathfrak{D}^j_{mj_0}(\varphi,\theta,-\varphi)p^{-i\lambda-1}\,|jm\rangle_{j_0\lambda}. \quad (3.19)$$

Using the orthogonality and completeness relations of the above coefficients, we obtain

$$|jm\rangle_{j_0\lambda} = \frac{1}{2\pi}\int\frac{d^3p}{p}(i)^j\left[\frac{(j-i\lambda)!}{(j+i\lambda)!}\right]^{\frac{1}{2}} \times (2j+1)^{\frac{1}{2}}\mathfrak{D}^{j*}_{mj_0}(\varphi,\theta,-\varphi)p^{i\lambda-1}\,|\mathbf{p}\rangle_{j_0}. \quad (3.20)$$

The transforms corresponding to the other three solutions for $\mu = 0$ can, of course, be calculated in an analogous fashion.

(*i*) $\mu > 0$

For $\mu > 0$, similar techniques can again be employed. But since the result has been given by Joos,[1] we content ourselves with briefly indicating how the transforms in question imply exactly the matrix elements of Sec. 1; it is sufficient of course to consider P_0 only.

For the case of zero spin (with $j_0 = 0$), the task is quite simple. We have only to utilize the recurrence relation for the Legendre functions:

$$(2\nu+1)zP^\mu_\nu(z) = (\nu-\mu+1)P^\mu_{\nu+1}(z) + (\nu+\mu)P^\mu_{\nu-1}(z). \quad (3.21)$$

Hence, assuming this result, we pass on to the case of nonzero spin. Here Joos's technique consists in expressing the coefficients (or the wavefunctions) *in the spinor representation* in the form

$$\{p, \underline{A} \mid mj, \lambda j_0) = \sum_{m_r, r}(s\underline{A}, rm_r \mid jm)\psi_{m_r,r,i\bar\lambda}a_{rj}(\lambda, j_0) \quad (3.22)$$

where $\psi_{\bar\lambda}$ behaves like a spin-zero wavefunction with

$$\bar\lambda = \lambda - ij_0 \quad (3.23)$$

and, moreover, due to a change of normalization involved in the definition, the matrix elements of P^0 are now

$$P_0\psi_{m_r,r,i\bar\lambda} = \frac{\mu}{2}\psi_{m_r,r,i(\bar\lambda-i)} + \frac{(i\bar\lambda+r)(i\bar\lambda-r-1)}{2i\bar\lambda(i\bar\lambda-1)}\psi_{m_r,r,i(\bar\lambda+i)}. \quad (3.24)$$

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[15] J. S. Lomont and H. E. Moses, J. Math. Phys. 3, 405, 1962; G. Berendt Preprint, Mathematisches Institut, Freie Universität Berlin, 1966.

We note that the coefficient $a_{r,j}(\lambda, j_0)$ determined by Joos can be expressed as

$$a_{r,j}(\lambda, j_0) = (-i)^{r+j_0-j}(2r+1)^{\frac{1}{2}}\begin{Bmatrix} s & j_2 & j_1 \\ j_2 & j & r \end{Bmatrix} C_{r,j}(\lambda, j_0), \tag{3.25}$$

where the $6j$ symbol involves the parameters

$$j_1 = \tfrac{1}{2}(j_0 - i\lambda - 1), \quad j_2 = -\tfrac{1}{2}(j_0 + i\lambda + 1)$$

or

$$j_1 + j_2 + 1 = -i\lambda, \quad j_1 - j_2 = j_0, \tag{3.26}$$

and

$$C_{r,j}(\lambda, j_0) = \frac{1}{M}\left[\frac{2(i\lambda + j_0)!\,(-i\lambda + j_0)!\,(i\lambda - j_0)!\,(-i\lambda - j_0 - r - 1)!\,(r - j_0 - i\lambda)!}{(-i\lambda - j_0)!\,(-i\lambda - s - 1)!\,(-j - i\lambda - 1)!\,(j + i\lambda)!\,(s + i\lambda)!}\right]^{\frac{1}{2}}. \tag{3.27}$$

The introduction of $6 - j$ symbols with complex coefficients should be considered merely as a formal way of conveniently exploiting the relevant recursion relations.

Now it is to be remarked that, while so far as $\psi_{\bar{\lambda}}$ alone is concerned, we cannot distinguish between the two ways of varying $\bar{\lambda}$, such as, for example

$$\bar{\lambda} \to \bar{\lambda} + i\begin{cases} \text{for} \quad \lambda \to \lambda + i; & j_1 \to j_1 + \frac{1}{2}, j_2 \to j_2 + \frac{1}{2}, \\ \text{or} \quad j_0 \to j_0 - 1: & j_1 \to j_1 - \frac{1}{2}, j_2 \to j_2 + \frac{1}{2}. \end{cases} \tag{3.28}$$

This is no longer the case when $a_{rj}(\lambda, j_0)$ is taken into account. Thus, we finally obtain all four matrix elements of P^0 given in Sec. 1.

In computing the matrix elements in this fashion, it is helpful to note the following relations:

$$\begin{aligned}\begin{Bmatrix} s & j_2 & j_1 \\ j_2 & j & r \end{Bmatrix} &= \frac{[(s + i\lambda)(s - i\lambda + 1)(j + i\lambda)(j - i\lambda + 1)]^{\frac{1}{2}}}{(-j_0 + i\lambda)[(r - j_0 - i\lambda + 1)(-j_0 - i\lambda - r)]^{\frac{1}{2}}}\begin{Bmatrix} s & j_2 + \frac{1}{2} & j_1 + \frac{1}{2} \\ j_2 + \frac{1}{2} & j & r \end{Bmatrix} \\ &\quad - \frac{[(s + j_0)(s - j_0 + 1)(j + j_0)(j - j_0 + 1)]^{\frac{1}{2}}}{(-j_0 + i\lambda)[(r - j_0 - i\lambda + 1)(-j_0 - i\lambda - r)]^{\frac{1}{2}}}\begin{Bmatrix} s & j_2 + \frac{1}{2} & j_1 - \frac{1}{2} \\ j_2 + \frac{1}{2} & j & r \end{Bmatrix},\end{aligned} \tag{3.28'}$$

$$\frac{C_{r,j}(\lambda, j_0)}{C_{r,j}(\lambda + i, j_0)} = \left[\frac{(i\lambda + j_0)(i\lambda - j_0)(-i\lambda + 1 - j_0)}{(j_0 - i\lambda + 1)(-i\lambda - j_0 - r)(r - j_0 - i\lambda + 1)}\right]^{\frac{1}{2}}, \tag{3.29}$$

and

$$\frac{C_{r,j}(\lambda, j_0)}{C_{r,j}(\lambda, j_0 - 1)} = -i\left[\frac{(i\lambda + j_0)(-i\lambda - j_0 + 1)(-i\lambda + j_0)}{(i\lambda - j_0 + 1)(r - j_0 - i\lambda + 1)(-i\lambda - j_0 - r)}\right]^{\frac{1}{2}}. \tag{3.30}$$

Finally, let us note the following point about the domains of variation of the parameters j_0 and λ. For $\mu = 0$, when j_0 (helicity) is fixed, we have to integrate over $-\infty < \lambda < \infty$. But for $\mu > 0$, exploiting the equivalence of the irreducible components $(j_0, \lambda) \to (-j_0, -\lambda)$, Joos[1] keeps only the values

$$j_0 = s, s-1, \cdots, \geq 0, \quad -\infty < \lambda < \infty, \tag{3.31}$$

whereas Ref. 13 retains the values

$$j_0 = -s, \cdots, s; \quad 0 \leq \lambda < \infty. \tag{3.32}$$

4. THE 4-VECTOR G AND DEFORMATIONS OF THE POINCARÉ ALGEBRA

Let

$$G_\mu = \tfrac{1}{2}(P^\nu M_{\nu\mu} + M_{\nu\mu}P^\nu) \tag{4.1}$$

$$= -i[P_\mu, \tfrac{1}{2}(\mathbf{N}^2 - \mathbf{M}^2)]. \tag{4.2}$$

Then, for example,

$$\begin{aligned} G_0\,|jm\rangle_{j_0\lambda} = -\frac{i}{2}[&(2j_0 + 1)\,{}_{j_0+1}\langle P_0\rangle_{j_0}\,|jm\rangle_{j_0+1\lambda} \\ &- (2j_0 - 1)\,{}_{j_0-1}\langle P_0\rangle_{j_0}\,|jm\rangle_{j_0-1\lambda} \\ &- (2i\lambda - 1)\,{}_{\lambda+i}\langle P_0\rangle_\lambda\,|jm\rangle_{j_0\lambda+i} \\ &+ (2i\lambda + 1)\,{}_{\lambda-i}\langle P_0\rangle_\lambda\,|jm\rangle_{j_0\lambda-i}] \end{aligned}$$

$$({}_{j_0+1}\langle P_0\rangle_{j_0} \equiv {}_{j_0+1\lambda}\langle jm|\,P_0\,|jm\rangle_{j_0\lambda}\ \text{etc.}). \tag{4.3}$$

Similarly, we can write down the matrix elements of $\mathbf{G}$ also.

We note the following relations involving G_μ:

$$[G_\mu, G_\nu] = iP^2M_{\mu\nu}, \quad [G_\mu, P_\nu] = i(P^2g_{\mu\nu} - P_\mu P_\nu),$$
$$[G_\mu, W_\nu] = -iP_\mu W_\nu, \tag{4.4}$$

and

$$[G^2 - W^2 + P^2(\mathbf{N}^2 - \mathbf{M}^2)] = \tfrac{9}{4}P^2. \tag{4.5}$$

Let

$$\begin{aligned} G'_\mu &= -i[G_\mu, \mathbf{N}\cdot\mathbf{M}] = -i[W_\mu, \tfrac{1}{2}(\mathbf{N}^2 - \mathbf{M}^2)] \\ &= \tfrac{1}{2}(P_\mu\mathbf{N}\cdot\mathbf{M} + \mathbf{N}\cdot\mathbf{M}P_\mu) \end{aligned} \tag{4.6}$$

and hence

$$\{G'^2 - 4P^2(\mathbf{N}\cdot\mathbf{M})^2\} = \tfrac{1}{4}W^2. \tag{4.7}$$

As is fairly well known (see, for example, Ref. 8), G_μ (suitably normalized) can be combined with P_μ in order to generate (along with $M_{\mu\nu}$) a representation of the de Sitter group, starting with an irreducible representation of the Poincaré group *with nonzero rest mass.*

The matrix elements and the Casimir operators of the representation thus obtained are given immediately through the formulas (4.3)–(4.7). In fact, a construction such as Eq. (4.2), where the modified "translation" operators are defined through a commutator with Casimir operators of the homogeneous subgroup, is sometimes termed "Gell-Mann formula."[9,16] However, it brings about a deformation of the original algebra (that of Poincaré group) in our case only when the "mass" $\mu \neq 0$. For $\mu = 0$, the original algebra is stable with respect to such a construction.

In order to treat the two cases together, let us define

$$P'_\mu = \epsilon_1 P_\mu + \epsilon_2 G_\mu. \tag{4.8}$$

Then

$$[P'_\mu, P'_\nu] = \epsilon_2^2[G_\mu, G_\nu] = i\epsilon_2^2 P^2 M_{\mu\nu} \tag{4.9}$$

and

$$P'^2 = \epsilon_1^2 P^2 + \epsilon_2^2 G^2. \tag{4.10}$$

Again, defining

$$W'_\mu = -i[P'_\mu, \mathbf{N}\cdot\mathbf{M}], \tag{4.11}$$

$$W'^2 = \epsilon_1^2 W^2 + \epsilon_2^2 G'^2, \tag{4.12}$$

where W^2 is given by Eqs. (2.25) or (2.39) and G'^2 by Eq. (4.7).

For $\mu > 0$, the usual procedure[8,17] is to take

$$\epsilon_1 = 1, \quad \epsilon_2 = \epsilon/\sqrt{p^2}, \tag{4.13}$$

where ϵ is interpreted as the "curvature" corresponding to the noncommuting "translations."

Let us now consider the case $\mu = 0$. For the case of discrete spin, it is evident that $(P'_\mu, M_{\mu\nu})$ generates once again a Poincaré algebra corresponding to zero mass and discrete spin.

For the case of continuous spin, Eq. (2.39),

$$P^2 = 0, \quad W^2 = -\tfrac{1}{4}\zeta^2 I, \tag{4.14}$$

we note that in P'^2 the parameter ζ^2 plays a "masslike" role.

In fact, now we have

$$[P'_\mu, P'_\nu] = 0, \tag{4.15}$$

$$P'^2 = -\epsilon_2^2\zeta^2/4, \tag{4.16}$$

$$W'^2 = -\epsilon_1^2\zeta^2/4 - \epsilon_2^2\zeta^2/16. \tag{4.17}$$

Thus for ζ real, P'_μ is spacelike for real ϵ_2.

[16] R. Hermann, Commun. Math. Phys. **2**, 155 (1966).
[17] C. Fronsdal, Rev. Mod. Phys. **37**, 221 (1965).

Again,

$$W'^2 = -P'^2\left[-\left(\frac{\epsilon_1^2}{\epsilon_2^2} + \frac{1}{4}\right)\right]$$
$$= -P'^2 s(s+1) \tag{4.18}$$

for

$$\epsilon_1^2/(-\epsilon_2^2) = \tfrac{1}{4}(2s+1)^2. \tag{4.19}$$

Thus for ϵ_1 fixed and

$$(-\epsilon_2^2) = \frac{4\epsilon_1^2}{(2s+2)^2}, \tag{4.20}$$

$$P'^2 = \frac{4\epsilon_1^2\zeta^2}{(2s+1)^2}, \tag{4.21}$$

we get a descending mass spectrum (compare Ref. 2, p. 340, and Ref. 11).

In order to get an ascending spectrum we are obliged to make ϵ_1 vary suitably with spin, say, for example, by putting

$$\epsilon_1^2 = \xi^2((2s+1)^2(s+1)). \tag{4.22}$$

5. DISCUSSION OF THE RESULTS

Discussion of the Formalism: Generalized Eigenvectors[18]*:* This part is not specific of the very problem which is studied here, and does not contain new results. It is rather concerned with a discussion of the formalism conventionally used (in that context see also Ref. 19), and with an interpretation of the properties of the spherical functions.

Let us recall first the definition of a generalized eigenvector.

Definition[18]*:* A generalized eigenvector of the operator A, defined on a linear topological space Φ, is a linear functional F on Φ such that

$$\forall \varphi \epsilon \Phi, \quad F(A\varphi) = \lambda F(\varphi). \tag{5.1}$$

λ is the corresponding eigenvalue.

A standard example: the eigenvectors $|p_\mu\rangle$ of the translation operators P_μ. On the space $\Phi = \varphi_4$, the generators $P_\mu = 1/i(d/dx_\mu)$ have as eigenvectors the functions $F = e^{ip_\mu\cdot x^\mu}$.

These functions do not belong to φ_4, but may be considered as belonging to the dual space φ'_4. Indeed, to each function $\varphi(x\mu)\epsilon\varphi_4$ corresponds a "scalar":

$$F(\varphi) = \tilde{\varphi}(p_\mu) = \frac{1}{(2\pi)^4}\int \varphi(x_\mu)e^{ip_\mu\cdot x\mu}\, dx_\mu,$$

which is the Fourier transform of $\varphi(x_\mu)$; Eq. (5.1) is obviously satisfied.

[18] Gel'fand-Chilov, *Les distributions tome 4* (Dunod Cie., Paris, 1964), Chap. I: Rigged Hilbert Spaces.
[19] J. P. Antoine, "General Dirac Formalism," preprint: University of Louvain. (to be published in Math. Phys.).

A meaning can also be given to the notion of an "orthogonal basis" for such generalized vectors.

Definition[18]*:* The set of generalized eigenfunctions is *complete*, if each element of Φ can be expanded in terms of the eigenfunctions.

For instance, each function $\varphi(x)$ of φ_4 can be expanded in terms of $e^{-ip\cdot x}$ by the integral

$$\varphi(x) = \frac{1}{(2\pi)^4}\int \tilde{\varphi}(p)e^{-ip\cdot x}d_4p \tag{5.2}$$

and the set of generalized eigenfunctions $e^{ip\cdot x}$ is complete.

Definition[20]*:* This set is *orthogonal*, if we have a completeness relation analogous to the Plancherel formula.

In our example,

$$\delta(p_\mu) = \frac{1}{(2\pi)^4}\int_{\mathbb{R}} e^{ip_\mu\cdot x_\mu}\,dx_\mu . \tag{5.3}$$

Generalized basis for the homogeneous Lorentz algebra "adapted" to a decomposition of a representation $[m, s]$ of the Poincaré algebra: As a consequence of a general result,[21] each strongly continuous unitary representation of the Poincaré group in the Hilbert space

$$\mathcal{H} = L^2\left(\frac{d_3\mathbf{p}}{2p_0}\right)$$

leads to a representation of the algebra, where all the generators are defined on a common domain, dense in $\mathcal{H}$. Let us call that domain Φ (the generators are selfadjoint on Φ).

The basis "adapted" to a decomposition of a representation $[m, 0]$ is given by the functions

$$F_{\lambda,j,m}(\mathbf{p}) = N_1\frac{1}{|\mathbf{p}|^{\frac12}}\,Y_j^m\left(\frac{\mathbf{p}}{|\mathbf{p}|}\right)p_{i\lambda-\frac12}^{-j-\frac12}\left(\frac{|p_0|}{M}\right)$$

where N_1 is a coefficient given in Ref. 1.

It was proved by Joos[1] that these functions are eigenfunctions of M^2, M_3, and F, with

$$F = -\tfrac14 M_{\mu\nu}M^{\mu\nu}$$

[the other Casimir of $SL(2, C)$: $G = \frac14\epsilon^{\mu\nu\varphi\sigma}M_{\mu\nu}M_{\varphi\sigma}$ being zero in that case], with the corresponding eigenvalues $j(j+1)$, m, and $(1+\lambda^2)/2$.

These functions are not square-integrable, and do not belong to the Hilbert space $\mathcal{H}$ (or to Φ, *a fortiori*). But, exactly as in the translation case, they may be considered as linear functionals on Φ by means of the transformation

$$F_{\lambda,j,m}(\varphi) = \tilde{\varphi}_{j,m}(\lambda) = \int\frac{d\mathbf{p}}{2p_0}\,\varphi(\mathbf{p})F^{\lambda}_{\lambda,j,m}(\mathbf{p}). \tag{5.4}$$

[20] R. Hermann, "Analytic Continuation of Group Representations. V" (to be published in Comm. Math. Phys.)

[21] See, for instance, H. D. Doebner and O. Melsheimer: Trieste IC 66-97.

A quite similar transformation has been studied by Vilenkin[22] from which we can see that Eq. (5.4) is defined when $\varphi(\mathbf{p}) \in \Phi$.

The self-adjointness of the operators M^2, M_3, and F on Φ is sufficient to assure the relation (5.1).

The functions $F_{\lambda,j,m}(\mathbf{p})$ are then the generalized eigenvectors corresponding to a given representation $(i\lambda, j_0 = 0)$ of $SL(2, C)$.

Now as one would expect, the relation (5.4) is a kind of Fourier transform, and the relations (5.2) and (5.3) can also be generalized to that case. In fact,[1,22] the set of generalized eigenfunctions $F_{\lambda,j,m}(\mathbf{p})$ is complete and orthogonal:

$$\varphi(\mathbf{p}) = \sum_{j,m}\int_0^\infty F_{\lambda,j,m}(\mathbf{p})\tilde{\varphi}_{j,m}(\lambda)\,d\lambda \tag{5.5}$$

and

$$\|\varphi(\mathbf{p})\| = \|\tilde{\varphi}_{j,m}(\lambda)\|. \tag{5.6}$$

Here the norm $\|\tilde{\varphi}_{j,m}(\lambda)\|$ is defined by

$$\sum_{j,m}\int d\lambda\,|\tilde{\varphi}_{j,m}(\lambda)|^2).$$

Equation (5.6) results from the Plancherel formula

$$\int_0^\infty\sum_{j,m}d\lambda F_{\lambda,j,m}(\mathbf{p})F^*_{\lambda,j,m}(\mathbf{p}') = 2\,|p_0|\,\delta(\mathbf{p}-\mathbf{p}'). \tag{5.7}$$

We do not discuss here the case $j_0 \neq 0$ $(s \neq 0)$.

Lorentz basis adapted to a decomposition of a representation corresponding to a zero mass and an helicity j_0. This basis is formed by the functions

$$H_{\lambda,j_0,j,m}(\mathbf{p}) = N_2P_0^{-i\lambda-1}D^m_{j,j_0}(\hat{p}) \quad\text{with}\quad \mathbf{p}^2 = P_0^2$$

and N_2 is a coefficient given in the expression (3.13).

They are also generalized eigenvectors for the two Casimirs F and G of $SL(2, C)$, and they form a complete and orthogonal set:

$$(N_2)^2\sum_{j_0jm}\int_{-\infty}^{+\infty}d\lambda p_0^{-i\lambda-1}p_0'^{i\lambda-1}D^m_{jj_0}(\Omega)D^{*m}_{jj_0}(\Omega')$$
$$= \frac{2\pi}{p_0}\,\delta(\Omega-\Omega')\delta(p_0-p_0'), \tag{5.8}$$

$$\|\tilde{\psi}_{j,m,j_0}(\lambda)\| = \|\psi(\mathbf{p})\|, \tag{5.9}$$

where $\tilde{\psi}_{j,m,j_0}(\lambda)$ is defined exactly as in Eq. (5.5).

How do the translation operators act on such as generalized basis?

The transformations (5.4), (5.7), etc., establish an isometry between the space Φ_p dense in $L^2(d\mathbf{p}/2p_0)$, and a space Φ_λ dense in $L^2(\sum_{j_0jm} d\lambda)$. [The elements $\varphi_{j_0,j,m}(\lambda)$ or Φ_λ were denoted in Sec. 2 by $|j, m\rangle_{j_0,\lambda}$.]

The action of the generators P_μ on the generalized

[22] N. Ya. Vilenkin, Dan **118**, No. 2, 219 (1958).

eigenvectors $F_{\lambda,j,m}(\mathbf{p})$ is defined by duality

$$\int [p_\mu F_{\lambda,j,m}(\mathbf{p})]\varphi(\mathbf{p}) \frac{d\mathbf{p}}{2p_0} = \int F_{\lambda,j,m}(\mathbf{p})[p_\mu \varphi(\mathbf{p})] \frac{d\mathbf{p}}{2p_0} = [\widetilde{p_\mu \varphi(\mathbf{p})}]_{j,m}(\lambda). \qquad (5.10)$$

Equation (5.10) has always a meaning for $\varphi \in \Phi$ (it is a consequence of the stability of Φ_p under the multiplication by p_μ), so $p_\mu \cdot F_{\lambda,j,m}(\mathbf{p})$ belongs to Φ.

In the case of the zero-mass representations, the action of P_μ is simply a multiplication. The result is a new linear functional

$$P_0 H_{j_0,\lambda,j,m}(\mathbf{p}) = N_2 P_0^{-i\lambda} D^j_{mj_0}(\varphi, \theta, -\varphi) = H_{j_0,\lambda+i,j,m}(\mathbf{p}).$$

Applying P_0 successively n times, we obtain the linear functionals $H_{j_0,\lambda+ni,j,m}(\mathbf{p})$.

Some comments about the results: At least formally, the functions $H_{j_0,\lambda+i,j,m}(\mathbf{p})$ [or equivalently, of $F_{\lambda+i,j,m}(\mathbf{p})$] correspond to a nonunitary representation of $SL(2, C)$ characterized by j_0, $\lambda + i$, which is not present in the unitary reduction of the Poincaré representation restricted to $SL(2, C)$.

We have seen that these functions have a meaning as functionals of Φ'_p: it simply results from the way they are constructed, Eq. (5.10). It can also be seen after integration with a function $f_{j_0,j,m}(\lambda) \in \Phi_\lambda$, since one then recovers the usual expressions of the Poincaré representation in a basis of impulsion.

But we are still faced with the problem of understanding the results in terms of the decomposition of the Poincaré representation on the Lorentz subalgebra and, in particular, of understanding the appearance of nonunitary representations of $SL(2, C)$. How such nonunitary representations can appear has to be understood in the sense of analytic continuation of group representations (and not of matrix elements).

We did not find a satisfactory interpretation of this result (which seems relatively new in the physical literature[23,24] and completely absent in the mathematical). In fact, we have not succeeded in finding an integral formula—as Cauchy formula for example—which would have described the appearing nonunitary representations as a continuous superposition of unitary representations (in a distribution sense).

Finally, let us only emphasize one aspect of the problem: the representation of the Poincaré algebra of the domain Φ_λ (λ a real number) cannot be integrated to a global representation of the Poincaré group; it can only be partially integrated to a representation of the Lorentz group.

[23] N. Mukunda, Syracuse Univ. preprint; This very recent preprint contains a discussion of the zero mass case.

[24] J. G. Kuriyan, N. Mukunda, and E. C. G. Sudarshan, "Master Analytical Representations, etc.," Preprint, Syracuse University (1967).

This is easily seen by observing that the Nelson operator cannot be essentially selfadjoint on Φ_λ, Φ_λ being not even stable under its action (see Ref. 22 for more details).

To sum up:

(1) Formally, the Wigner–Eckart theorem is still valid. This is particularly clear in Sec. 2; the matrix elements of P_μ between two homogeneous Lorentz-group representation are of the following form:

$$_{j_0',\lambda'}\langle j'm' | P_\mu | jm\rangle_{j_0,\lambda} = C_\mu(jj'mm'; j_0j_0', \lambda\lambda') \langle j_0'\lambda' \| P \| j_0\lambda\rangle,$$

where $C_\mu(jj'mm'j_0j_0'\lambda\lambda')$ is the product of a Clebsch–Gordan coefficient of the rotation group with a coefficient which is characteristic of the Lorentz group representations involved.

For instance,

$$C_3(j, j+1, m, m, j_0j_0, \lambda, \lambda + i) = \left[\frac{(j+1)^2 - m^2}{(2j+3)(2j+1)}\right]^{\frac{1}{2}} [(j+1+i\lambda)(j+2-i\lambda)]^{\frac{1}{2}} \cdot \times \left[\left(\frac{-i}{j+1}\right)\left((j+1)^2 - j_0^2\right)^{\frac{1}{2}}\right].$$

The second coefficient is obtained from Eqs. (2.9) and (2.10).

In other words, it is equivalent to say that the product of the Lorentz representations

$$D^{0,2} \oplus D^{j_0,i\lambda}$$

has the same decomposition (at least in a distribution sense) as that in the case of the finite-dimensional representations, i.e.,

$$D^{0,2} \otimes D^{j_0,i\lambda} = D^{j_0,i\lambda+1} \oplus D^{j_0,i\lambda-1} \oplus D^{j_0+1,i\lambda} \oplus D^{j_0-1,i\lambda}.$$

In Gel'fand notation D^{p_1,p_2}: ($D^{0,2}$ corresponds to the usual $D^{\frac{1}{2},\frac{1}{2}}$.)

(2) In a very formal sense, the direct integral

$$\sum_{j_0} \int d\lambda D^{j_0,i\lambda}$$

of Lorentz representation is "equivalent" to a direct sum

$$\bigoplus_{n=-\infty}^{+\infty} D^{j_0+p,i\lambda+n}, \quad 0 < j_0 + p \le s,$$

of nonunitary representations of $SL(2, C)$ (since one recovers the same results after testing with suitable functions).

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Random Spin Systems: Some Rigorous Results*

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Several general results are obtained for a system of spins on a lattice in which the various lattice sites are occupied at random, and the spins, if present, interact via a general Heisenberg or Ising interaction decreasing sufficiently rapidly with distance. It is shown that the free energy per site exists in the limit of an infinite system, is a continuous function of concentration, and has the usual convexity (stability) properties. For Ising systems with interactions of finite range, the free energy is an analytic function of concentration and magnetic field for a suitable range of these variables. The random Ising ferromagnet on a square lattice (or simple cubic lattice) with nearest-neighbor interactions is shown to exhibit a spontaneous magnetization at sufficiently high concentrations and low temperatures.

1. INTRODUCTION

The problem of idealized Heisenberg or Ising ferromagnets on regular lattices in which certain sites, chosen at random, are vacant has been extensively studied in connection with the problem of ferromagnetism in quenched dilute alloys.[1] Whereas the physical properties of such alloys require a more complex analysis than was first thought to be the case, the model calculations have provided at least a qualitative guide in interpreting experimental data.[2] Investigations of statistical properties of a random spin system should also yield insight into the effects of impurities on phase transitions and critical points, both in magnetic and nonmagnetic systems.[3]

We shall discuss here, from a rigorous point of view, certain mathematical properties of such systems associated with taking the infinite volume or "thermodynamic" limit. In essence, we wish to extend to at least a certain class of random systems some of the results already known to hold for regular systems. The mathematical techniques are unfortunately not of much value in making good estimates of thermodynamic properties, transition temperatures, critical concentrations, etc. Nonetheless, in the absence of soluble models (except in one dimension), general or "rigorous" results may prove valuable as a guide to intuition and a check on the consistency of approximate calculations.

An outline of our paper is as follows. In Sec. 2, the random spin problem is defined and some notation essential to further developments is introduced. Section 3 contains a proof of the infinite-volume limit of the free energy under fairly general conditions on the Hamiltonian. In Sec. 4 we show that an Ising ferromagnet with nearest-neighbor interactions on a square lattice will exhibit a spontaneous magnetization over a certain range of temperatures and concentration. The Ising-model free energy for a system with interactions of finite range is an analytic function of concentration and magnetic field for suitable ranges of these variables; the proof is found in Sec. 5. Some additional results are stated without proof in Sec. 6.

In the main, our procedures are simply an adoption to the problem at hand of mathematical techniques already published and discussed at length in several different papers. For this reason our proofs are somewhat abbreviated and certain steps are omitted in the interests of brevity; we have tried to include a complete discussion of the modifications required and difficulties encountered in applying the "well-known" methods to random systems.

2. NOTATIONS AND DEFINITIONS

A finite system or crystal Ω (which we shall usually assume has a simple shape, for instance, a cube) consists of N_Ω sites lying on a regular lattice. (We

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[1] H. Sato, A. Arrott, and R. Kikuchi, J. Phys. Chem. Solids **10**, 19 (1959); R. J. Elliott, *ibid.* **16**, 165 (1960); J. S. Smart, *ibid.* **16**, 169 (1960); M. Coopersmith and R. Brout, *ibid.* **17**, 254 (1961); R. J. Elliott and B. R. Heap, Proc. Roy. Soc. (London) **A265**, 264 (1962); B. R. Heap, Proc. Phys. Soc. (London) **82**, 252 (1963); D. H. Lyons, Phys. Rev. **128**, 2022 (1962); G. S. Rushbrooke and D. J. Morgan, Molecular Phys. **4**, 1, 291 (1961); **6**, 477 (1963); P. J. Wojtowicz, *ibid.* **6**, 157 (1963); G. S. Rushbrooke, J. Math. Phys. **5**, 1106 (1964); T. Morita, *ibid.* **5**, 1401 (1964); R. Abe, Progr. Theoret. Phys. (Kyoto) **31**, 412 (1964); N. W. Dalton, C. Domb, and M. F. Sykes, Proc. Phys. Soc. (London) **83**, 496 (1964); S. Katsura and B. Tsujiyama, in *Critical Phenomena*, M. S. Green and J. V. Sengers, Eds. (National Bureau of Standards, Washington, D.C., 1966), p. 219. See also Refs. 2–5 below.
[2] See, for example, M. W. Klein and R. Brout, Phys. Rev. **132**, 2412 (1963); C. E. Violet and R. J. Borg, *ibid.* **149**, 540 (1966).
[3] For some applications to nonmagnetic systems, see G. M. Bell, Proc. Phys. Soc. (London) **72**, 649 (1958).

shall omit the subscript on N when the system referred to is clear from context.) A *partition* is a subset θ of sites from Ω which are occupied by spins, the remaining sites being empty. The case $\theta = \Omega$, all sites occupied, we call a *regular* system as distinct from a *random* system for which, in general, θ is a proper subset.

For a given partition θ, we define a spin Hamiltonian

$$\mathcal{H}(\theta) = -2\sum_{i\in\theta}\sum_{j\in\theta} J_{ij}[S_i^z S_j^z + \gamma(S_i^x S_j^x + S_i^y S_j^y)] - 2\mu H \sum_{i\in\theta} S_i^z, \quad (2.1)$$

where $\mathbf{S}_i$ is the quantum-mechanical spin operator on the ith site; $J_{ij} = J_{ji}$ is the "exchange integral," equal to zero for $i = j$; μ is the magnetic moment; H, the external magnetic field. The x component of $\mathbf{S}_i$, S_i^x, has eigenvalues $S, S-1, S-2, \cdots, -S$, and likewise the y and z components; S, the "total spin quantum number," is a positive integer or half-odd integer. For $\gamma = 0$, we have an "Ising model" and, for $\gamma = 1$, a "Heisenberg model." The J_{ij} in (2.1) are regarded as functions only of the relative location of sites i and j and are independent of the partition θ. Note that the sums in (2.1) extend only over occupied sites.

The free energy F, defined by

$$e^{-\beta F} = \mathrm{Tr}\,[e^{-\beta\mathcal{H}}], \quad (2.2)$$

where $\beta = (kT)^{-1}$, the inverse temperature, is a function of θ through (2.1). If θ contains n sites, "Tr" stands for a trace over all the $(2S+1)^n$ possible states or *configurations* of the n spins located on the occupied sites. The magnetization M and entropy S for the partition in question may be found, as usual, by differentiation:

$$M(\theta) = -\left(\frac{\partial F(\theta)}{\partial H}\right)_T, \quad (2.3)$$

$$S(\theta) = -\left(\frac{\partial F(\theta)}{\partial T}\right)_H. \quad (2.4)$$

In the *random spin problem* or *random impurity problem* one assigns to each partition a probability $P(\theta)$ and defines the free energy for the crystal as a whole by

$$F_\Omega = \sum_\theta P(\theta)F(\theta). \quad (2.5)$$

We shall henceforth assume that $P(\theta)$ is independent of temperature and magnetic field. Physically, this is the assumption that the random "impurities" are frozen in position; Brout[4] has pointed out that it should not be an unreasonable model of a real magnetic crystal in which the motion of various impurities is relatively slow and the time required for them to come to some ultimate equilibrium is long compared with magnetic relaxation in the spin system itself and the time scale of ordinary magnetic experiments. (It is also possible to define models in which "impurity" equilibrium as well as magnetic equilibrium is assumed[5]; we shall not discuss these here.)

In comparison with regular magnetic systems, random systems (as we have defined them) possess additional complexity through the existence of two kinds of average. There is the ordinary thermal average of an operator $\mathcal{O}$ in some partition θ defined by

$$\langle\mathcal{O}\rangle_\theta = \frac{\mathrm{Tr}\,[\mathcal{O}e^{-\beta\mathcal{H}(\theta)}]}{\mathrm{Tr}\,[e^{-\beta\mathcal{H}(\theta)}]} \quad (2.6)$$

and, in addition, the average over partitions of some function $g(\theta)$ (which could, for example, be $\langle\mathcal{O}\rangle_\theta$):

$$\langle\langle g\rangle\rangle_\Omega = \sum_\theta P(\theta)g(\theta). \quad (2.7)$$

We shall consider two possible forms for $P(\theta)$:

(a) For a fixed value of n,

$$P = \begin{cases} \binom{N}{n}^{-1}, & \text{if } \theta \text{ contains } n \text{ sites,} \\ 0, & \text{otherwise.} \end{cases} \quad (2.8)$$

(b) Choose some p between 0 and 1, and let $q = 1 - p$. When θ contains n sites of a total of N in Ω,

$$P(\theta) = p^n q^{N-n}. \quad (2.9)$$

Clearly, (b) is equivalent to the assumption that each individual site is occupied with probability p and vacant with probability q, and the occupation of different sites is statistically independent. It is an assumption frequently made in random spin calculations. The relationship of (a) to (b) is analogous to the relationship between canonical and grand canonical "ensembles" in the statistical mechanics of regular systems. The analogy should not be pressed too far, and the proof of asymptotic equivalence of (a) and (b) for large crystals (Sec. 3) is a bit different from standard arguments relating canonical and grand canonical ensembles.

3. EXISTENCE AND PROPERTIES OF THE ASYMPTOTIC FREE ENERGY

Let $P(\theta)$ be given by (2.9). The free energy for a given crystal Ω is [see Eq. (2.5)]

$$N_\Omega f_\Omega(p) = F_\Omega(p) = \sum_{\theta\subset\Omega} F(\theta)p^{\nu(\theta)}q^{N-\nu(\theta)}, \quad (3.1)$$

[4] R. Brout, Phys. Rev. **115**, 824 (1959); R. R. Mazo, J. Chem. Phys. **39**, 1224 (1963).

[5] For example, G. M. Bell and W. M. Fairbairn, Phil. Mag. **6**, 907 (1961); G. M. Bell and D. A. Lavis, *ibid.* **11**, 937 (1965); I. Syozi and S. Miyazima, Progr. Theoret. Phys. (Kyoto) **36**, 1083 (1966).

where $\nu(\theta)$ is the number of occupied sites in Ω. Provided certain conditions are satisfied by the Hamiltonian (2.1) and by a sequence of crystals Ω of increasing volume, we shall show that

$$f = \lim_{N_\Omega \to \infty} f_\Omega \qquad (3.2)$$

exists.

In several papers, the existence of the limit (3.2) or its analogs has been demonstrated for regular systems.[6,7] The physical idea which underlies all the proofs (and is not always clear in the thicket of mathematical detail) is quite simple: If a large macroscopic system is split into a number of macroscopic parts, its free energy is the sum of the free energies of the different parts plus correction terms due to the interactions of adjacent systems across their common boundary. The correction terms, being (roughly) proportional to surface areas, become negligible for large systems as the surface-to-volume ratio approaches zero.

Consider two systems Ω_1 and Ω_2 with free energies F_1 and F_2 which together constitute a total system Ω with free energy F. The totality of sites in a pair of partitions θ_1, θ_2 in Ω_1, Ω_2 clearly constitute a partition θ_{12} for Ω with a probability

$$P(\theta_{12}) = P(\theta_1)P(\theta_2). \qquad (3.3)$$

We may write

$$\mathcal{H}(\theta_{12}) = \mathcal{H}(\theta_1) + \mathcal{H}(\theta_2) + \mathcal{H}^*(\theta_1, \theta_2), \qquad (3.4)$$

where $\mathcal{H}^*$ contains the terms in the double sum (2.1) for which i lies in θ_1 and j in θ_2 or vice versa; that is, it represents a "surface term" involving in a significant way only spins fairly close to the boundary separating Ω_1 and Ω_2.

In Ref. 6 it is shown that if $\mathcal{H}_A$ and $\mathcal{H}_B$ are two Hamiltonians in the same vector space (note that in the present context a Hamiltonian is a *finite*-dimensional Hermitian matrix), the associated free energies defined by (2.2) satisfy an inequality

$$|F(\mathcal{H}_A) - F(\mathcal{H}_B)| \le |\mathcal{H}_A - \mathcal{H}_B|, \qquad (3.5)$$

where the norm $|\mathcal{H}|$ of a Hermitian operator $\mathcal{H}$ is simply the largest of the absolute values of its eigenvalues. We may apply (3.5) to (3.4) by letting $\mathcal{H}_A$ be $\mathcal{H}(\theta_{12})$ and $\mathcal{H}_B$ be $\mathcal{H}(\theta_1) + \mathcal{H}(\theta_2)$; note that $\mathcal{H}_B$ is the Hamiltonian for two noninteracting systems. It follows that

$$|F(\theta_{12}) - [F(\theta_1) + F(\theta_2)]| \le |\mathcal{H}^*(\theta_1, \theta_2)|. \qquad (3.6)$$

To obtain a bound independent of θ_{12} we note that $|\mathcal{H}^*|$ is less than the sum of the norms of the individual terms making up $\mathcal{H}^*$, which sum will be a maximum for that partition in which all sites are occupied. That is, $|\mathcal{H}^*(\theta_1\theta_2)|$ is bounded by the corresponding bound used in Ref. 6 for the case of a perfect crystal—all sites occupied in both systems 1 and 2,

$$|\mathcal{H}^*(\theta_{12})| \le A_{12}. \qquad (3.7)$$

Finally, noting that

$$\sum_{\theta_{12}} P(\theta_{12})[F(\theta_{12}) - F(\theta_1) - F(\theta_2)] = F - F_1 - F_2, \qquad (3.8)$$

we obtain with the help of (3.6) and (3.7)

$$|F - F_1 - F_2| \le A_{12}. \qquad (3.9)$$

The inequality (3.9) is the rigorous counterpart of the intuitive arguments mentioned earlier. Its generalization to the case of several systems placed in contact is immediate, and once it has been obtained the proof of an infinite volume limit reduces to an exercise in geometry (see Refs. 6, 7) which we shall not repeat here. The final result is embodied in the following.

Theorem 1: The limit (3.2) exists provided two conditions are satisfied.

1. The J_{ij} in (2.1) depend only on the relative positions of sites i and j (translational invariance) and possess a bound

$$|J_{ij}| \le C/r_{ij}^{d+\epsilon}, \qquad (3.10)$$

where r_{ij} is the distance between sites i and j, d is the dimensionality of the lattice, C and ϵ are strictly positive numbers (it is essential to have $\epsilon > 0$), independent of the choice of i and j.

2. The sequence Ω is of crystals with sufficiently regular shape; for example, a d-dimensional parallelepiped with all d edges increasing to infinity.

Note that the thermodynamic limit f as a function of H, T is convex upwards, or concave, in both variables together. That is, if two points of the $f(H, T)$ surface are joined by a chord, the chord lies entirely on or below the surface. This follows from the observation that for any finite crystal Ω and specific partition θ, $F_\Omega(\theta, H, T)$ is concave,[6] while averages (2.5) and limits (3.2) of concave functions inherit the same property. The convexity (concavity) property is equivalent to the "stability" conditions of positive susceptibilities and heat capacities. It also implies that f is a continuous function of both H and T in the range $0 < T < \infty$, $-\infty < H < \infty$.

The foregoing results have all been established with the assumption that p, the fraction of occupied sites,

[6] R. B. Griffiths, J. Math. Phys. 5, 1215 (1964).

[7] The arguments for spin systems are closely analogous to those used for classical and quantum fluids; see C. N. Yang and T. D. Lee, Phys. Rev. 87, 404 (1952); D. Ruelle, Helv. Phys. Acta 36, 183, 789 (1963); M. E. Fisher, Arch Rational Mech. Anal. 17, 377 (1964).

is held constant. We now wish to examine the dependence of f in (3.1) on p, and also the "microcanonical" probability (2.8). In connection with the latter let us introduce the quantity

$$\hat{F}_\Omega(n) = \binom{N}{n}^{-1} \sum_{\nu(\theta)=n} F(\theta), \tag{3.11}$$

for a crystal Ω with N sites, where the summation is over all partitions θ having $\nu(\theta) = n$ sites. We shall need the following result:

Theorem 2: Provided the temperature is less than infinity, and n and m lie between 0 and N, there exists a finite constant D, depending on the Hamiltonian (2.1) and the temperature, but independent of Ω, N, n, and m, such that

$$|\hat{F}(m) - \hat{F}(n)| \leq D\,|n - m|. \tag{3.12}$$

The proof of this theorem depends on two lemmas, the first of which is an almost obvious combinatorial result. Given a partition θ with n occupied sites, let $\theta_1, \theta_2, \cdots, \theta_{N-n}$ be the $N - n$ distinct partitions with $n + 1$ occupied sites obtained by adding to θ one additional site from Ω.

Lemma 1: Let g be any function defined on the partitions. Then

$$\sum_{\nu(\theta)=n+1} g(\theta) = \frac{1}{n+1} \sum_{\nu(\theta)=n} \sum_{j=1}^{N-n} g(\theta_j). \tag{3.13}$$

This result merely expresses the fact that every partition containing $n + 1$ sites appears in the double sum on the right side of (3.13) precisely $n + 1$ times, since this is the number of distinct partitions containing n sites from which it can be derived by the addition of one more site.

Lemma 2: Let θ_k be a partition containing $n + 1$ sites obtained from a partition θ of n sites by the addition of one site, say the site l. Then

$$|F(\theta_k) + \beta^{-1} \ln (2S + 1) - F(\theta)| \leq h, \tag{3.14}$$

where

$$h = 4 \sum_j |J_{lj}[S_l^z S_j^z + \gamma(S_l^x S_j^x + S_l^y S_j^y)]| + 2\mu H\,|S_l^z|. \tag{3.15}$$

The sum over j in (3.15) extends over all lattice sites; (3.10) insures its convergence (note that $J_{ll} = 0$).

Proof: (Theorem 2) The comparison of $F(\theta)$ and $F(\theta_k)$ is conveniently carried out in two steps. The first is to add the site l to θ_l but set all the terms in (2.1) which involve $\mathbf{S}_l$ equal to zero. The resulting Hamiltonian $\mathcal{H}'(\theta_k)$ is formally identical to $\mathcal{H}(\theta)$ but is defined in a larger space, the $(2S + 1)^{n+1}$ configurations of θ_k. Hence with each eigenstate of $\mathcal{H}(\theta)$ are associated $2S + 1$ eigenstates of $\mathcal{H}'(\theta_k)$ having the same eigenvalue, and the corresponding free energies are related by

$$F'(\theta_k) = F(\theta) - \beta^{-1} \ln (2S + 1). \tag{3.16}$$

As a second step, we employ (3.5)

$$|F(\theta_k) - F'(\theta_k)| \leq |\mathcal{H}(\theta_k) - \mathcal{H}'(\theta_k)|, \tag{3.17}$$

and note that the two Hamiltonians differ in that one lacks all terms involving $\mathbf{S}_l$. The right side of (3.17) is bounded by the sum of the norms of these terms, which in turn is bounded by (3.15). Thus (3.16) and (3.17) together imply (3.14).

If we define

$$D = h + \beta^{-1} \ln (2S + 1), \tag{3.18}$$

(3.14) implies that

$$\left| \sum_{j=1}^{N-n} F(\theta_j) - (N - n)F(\theta) \right| \leq (N - n)D. \tag{3.19}$$

If we insert this result in (3.13) (with g replaced by F) and use the definition (3.11), we obtain

$$|\hat{F}(n + 1) - \hat{F}(n)| \leq D, \tag{3.20}$$

from which (3.12) follows by an obvious iteration. This completes the proof of Theorem 2.

By combining (3.1) and (3.11) one obtains

$$F(p) = \sum_{n=0}^{N} b(n; N, p)\hat{F}(n), \tag{3.21}$$

where

$$b(n; N, p) = \binom{N}{n} p^n q^{N-n} \tag{3.22}$$

is the binomial distribution. Let n_p be the nearest integer to pN. The inequality (3.12) yields the bound

$$\begin{aligned} |F(p) - \hat{F}(n_p)| &\leq D \sum_n |n - n_p|\, b(n; N, p) \\ &\leq DA \sum_{|n-n_p| \leq A} b(n; N, p) + DN \sum_{|n-n_p|>A} b(n; N, p). \end{aligned} \tag{3.23}$$

By setting $A = N^{\frac{2}{3}}$ and employing an elementary estimate for the tails of the binomial distribution,[8] we find

$$|F(p) - \hat{F}(n_p)| \leq 0(DN^{\frac{2}{3}}). \tag{3.24}$$

Let $\hat{f}_\Omega(p)$ be defined by linear interpolation between successive points where Np is an integer, $0 \leq p \leq 1$, and at these points by the relation

$$\hat{f}_\Omega(p) = \hat{F}_\Omega(pN_\Omega)/N_\Omega. \tag{3.25}$$

[8] H. Cramer, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, N.J., 1946), p. 196.

Comparison with (3.1) and (3.24) shows that

$$|f_\Omega(p) - \hat{f}_\Omega(p)| \leq O(DN_\Omega^{-\frac{1}{3}}), \qquad (3.26)$$

which implies that f_Ω and $\hat{f}_\Omega$ converge to the same thermodynamic limit $f(p)$. Theorem 2 combined with (3.25), (3.26), and (3.2) tell us that

$$|f(p_1) - f(p_2)| \leq D\,|p_1 - p_2|, \qquad (3.27)$$

where D is a continuous function of T and H [see (3.18) and (3.15)]. Since $f(p, H, T)$ at fixed p is known to be a continuous function of H and T by convexity, (3.27) shows that it is a continuous function of all three variables together.

4. SPONTANEOUS MAGNETIZATION IN ISING FERROMAGNETS

A large amount of information is available on the phase transition and critical point behavior of Ising ferromagnets in two and three dimensions with nearest-neighbor interactions.[9] It would be interesting to know what effect the addition of random nonmagnetic impurities has on the phase transition. With the relatively unrefined tools available for an exact analysis, we are unable to make any statement about the modification of the critical point indices upon addition of impurities. However, we shall demonstrate, using the appropriate modifications of an argument originally due to Peierls,[10,11] that at sufficiently high concentrations and sufficiently low temperatures, a specific random Ising ferromagnet exhibits a spontaneous magnetization; i.e., as the magnetic field is reduced to zero from positive values, the bulk magnetization m, defined as

$$m = -\left(\frac{\partial f}{\partial H}\right)_{T,p}, \qquad (4.1)$$

approaches a positive limit. To be specific we shall consider a system with Hamiltonian

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - H \sum_i \sigma_i, \qquad (4.2)$$

where $\sigma_i = 2S_i^z = \pm 1$, the sum in (4.2) is over nearest-neighbor pairs of sites, and the constant J is positive. For a random system we make the obvious modifications in accordance with (2.1); the second sum in (4.2) extends only over occupied sites and the first over pairs of nearest-neighbor sites both of which are occupied.

If $\mathcal{N}_-$ is the number of "down" spins ($\sigma_i = -1$), $\mathcal{N}_+$ the number of "up" spins ($\sigma_i = +1$), the magnetization operator $\mathcal{M}$ is given by

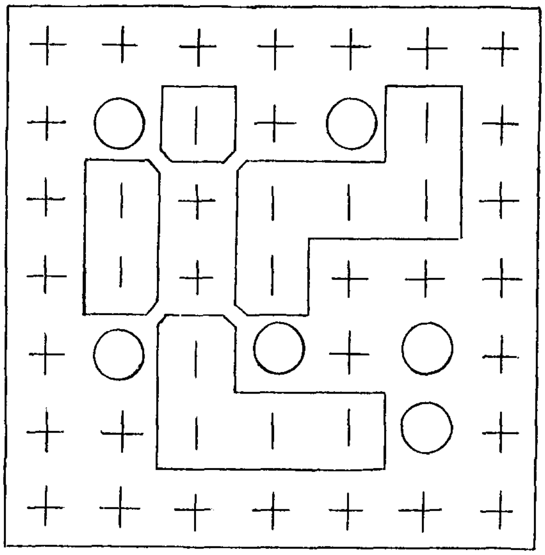

FIG. 1. A typical configuration on a square lattice of a random Ising system. The symbols +, − denote spins up and down, respectively, on occupied sites; 0 means the site is vacant. All boundary sites are occupied with + spins, and thus all − spins are enclosed within closed polygons (shown by the heavy lines).

$$\mathcal{M} = \mathcal{N}_+ - \mathcal{N}_-. \qquad (4.3)$$

We shall establish an inequality of the form

$$\langle\langle \mathcal{N}_- \rangle\rangle_\Omega \leq \tfrac{1}{2}(p - \epsilon)N_\Omega, \qquad (4.4)$$

with $\epsilon > 0$, for a series of crystals Ω with a special boundary condition but in zero magnetic field, $H = 0$ in (4.2). Here p is the fraction of sites occupied on the average, assuming $P(\theta)$ is given by (2.9). Since $\mathcal{N}_+ + \mathcal{N}_-$ is simply the number of occupied sites, with average value equal to pN_Ω, (4.4) implies that

$$\langle\langle \mathcal{M} \rangle\rangle_\Omega \geq \epsilon N_\Omega. \qquad (4.5)$$

In turn, (4.5) implies that

$$\lim_{H \to 0+} m(H) \geq \epsilon. \qquad (4.6)$$

The connection between (4.5) and (4.6) is a trifle subtle, and is discussed for regular systems in Ref. 12. Identical considerations hold for random systems.

We shall consider a sequence of square crystals on a square lattice with the following special boundary conditions: The spins on all occupied sites on the edges of the squares are subjected to a magnetic field $J > 0$; alternatively, the sites just outside each edge are occupied with Ising spins for which $\sigma_i = +1$ (Fig. 1). Given any partition θ and any configuration for this partition, it is possible to enclose all spins with $\sigma_i = -1$ inside borders—nonself-intersecting polygons consisting of lines passing midway between lattice sites, as shown in Fig. 1.

[9] M. E. Fisher, Rept. Progr. Phys. **30**, 615 (1967).
[10] R. Peierls, Proc. Cambridge Phil. Soc. **32**, 477 (1936).
[11] R. B. Griffiths, Phys. Rev. **136**, A437 (1964); see also R. L. Dobrushin, Teoriya Veroyatnostei Primeneniya **10**, 209 (1965) [Theory Probability Appl. **10**, 193 (1965)].
[12] R. B. Griffiths, Phys. Rev. **152**, 240 (1966).

A particular site is *adjacent* to a border B—to the exterior or interior of B depending on whether the site lies outside or inside B—and has a *coordination number* z relative to B provided B passes between the site and $z \geq 1$ of its nearest neighbor. The border B is *realized* and the operator X_B assigned the value 1 provided (i) every site adjacent to the interior of B is occupied by a spin with $\sigma = -1$ and (ii) every site adjacent to the exterior of B is either unoccupied or occupied by a spin with $\sigma = +1$. Otherwise X_B has the value 0. The requirement that a border B not intersect itself has the consequence that z does not exceed three, except when B surrounds only one site (Fig. 1).

For a particular border B (fixed in position and orientation) and a given partition θ, let $r(B, \theta, z)$ be the number of occupied sites of coordination number z, relative to B, adjacent to the exterior of B. Define

$$\eta(B, \theta) = \sum_z z r(B, \theta, z). \tag{4.7}$$

When $X_B = 1$, η is the number of segments of B lying between pairs of nearest-neighbor occupied sites with $\sigma_i = +1$ and -1 on the site exterior and interior to B, respectively. The argument of Ref. 11 shows that

$$\langle X_B \rangle_\theta \leq y^{\eta(B,\theta)}, \tag{4.8}$$

where

$$y = e^{-2\beta J}. \tag{4.9}$$

[In essence, (4.8) is obtained by noting that for every configuration in which B is realized there is another, obtained by reversing every spin interior to B, which is lower in energy by $2\eta J$.] Note that (4.8) is satisfied even for partitions θ in which B cannot be realized because sites adjacent to its interior are vacant—for these $\langle X_B \rangle$ is necessarily zero.

Next, average (4.8) over partitions:

$$\begin{aligned}\langle\langle X_B \rangle\rangle &\leq \sum_\theta P(\theta) y^{\eta(B,\theta)} \\ &= \sum_{r_1} \sum_{r_2} \sum_{r_3} P_B(r_1, r_2, r_3) y^{r_1 + 2r_2 + 3r_3},\end{aligned} \tag{4.10}$$

where $P_B(r_1, r_2, r_3)$ is the probability that precisely r_z occupied sites of coordination number z with respect to B are adjacent to the exterior of B. If k_z is the maximum possible value of r_z (given B), the value when all sites are occupied, then

$$P_B = \prod_{z=1}^{3} \binom{k_z}{r_z} p^{r_z} q^{k_z - r_z}. \tag{4.11}$$

The insertion of (4.11) in (4.10) yields the estimate

$$\begin{aligned}\langle\langle X_B \rangle\rangle &\leq (yp + q)^{k_1} (y^2 p + q)^{k_2} (y^3 p + q)^{k_3} \\ &\leq (y^3 p + q)^{b/3},\end{aligned} \tag{4.12}$$

where

$$b = \sum_{z=1}^{3} z k_z \tag{4.13}$$

is the length of the border B in units of the lattice constant. The final inequality in (4.12) is derived in the Appendix.

Since every minus spin is inside some realized border, and the number of occupied sites inside a border does not exceed $b^2/4$, we may write

$$\langle\langle \mathcal{N}_- \rangle\rangle \leq \sum_b \nu(b)(b^2/4)(y^3 p + q)^{b/3}. \tag{4.14}$$

If for $\nu(b)$, the number of different polygons of length b in a crystal Ω containing N_Ω sites, we use the generous estimate[13]

$$\nu(b) \leq 4 N_\Omega 3^{b-2}/b, \tag{4.15}$$

and carry out the sum (4.14) over the values $b = 4, 6, 8, \cdots$; the result is

$$\langle\langle \mathcal{N}_- \rangle\rangle \leq \tfrac{2}{9} K^2 \frac{2 - K}{(1 - K)^2} N_\Omega, \tag{4.16}$$

where

$$K = 9(y^3 p + q)^{\frac{2}{3}}. \tag{4.17}$$

Comparison with (4.4) shows that for sufficiently high concentrations and for y small enough (that is, at low enough temperatures), there will be a spontaneous magnetization. Due to the severe approximations involved, the estimate (4.16) is not very good. In particular, the concentration p must exceed approximately 0.985 before it will guarantee a spontaneous magnetization at a finite temperature. One actually expects that the concentration below which no spontaneous magnetization exists at any temperature will coincide with the critical concentration of the corresponding percolation problem,[14] that is, approximately 0.59. Nonetheless, our argument shows that there is a finite range of concentrations for which vacancies in the lattice do not alter the qualitative nature of the phase transitions, in that the bulk magnetization still shows a discontinuity at zero magnetic field.

An argument analogous to that given above can also be carried out for, say, a three-dimensional Ising model on a simple cubic lattice. Or one may apply a general inequality for correlations in Ising ferromagnets in order to obtain a proof for three dimensions directly from the knowledge that a spontaneous magnetization exists for the two-dimensional case.

5. ANALYTIC PROPERTIES OF THE FREE ENERGY

A regular Ising spin system is isomorphic to a lattice gas, if an "up" spin indicates the presence, and a

[13] See, for example, G. H. Wannier, *Elements of Solid State Theory* (Cambridge University Press, Cambridge, England, 1959), p. 105. Note that there is no choice left in choosing the final step in a border, so we have reduced Wannier's estimate by a factor of 3.

[14] M. F. Sykes and T. W. Essam, Phys. Rev. **133**, A310 (1964).

"down" spin the absence, of a molecule at a particular site. In the analogous isomorphism for a random Ising spin system, one has two types of vacant sites. For a given partition θ, there are sites outside θ where gas molecules cannot be found because there is (in the spin language) no spin on the site. One may think of these sites as occupied by fixed impurity molecules whose presence excludes gas molecules. Then there are sites which belong to θ but at which, for a particular configuration, the spin is "down." One may think of these as "vacuum" sites since there are other configurations (for the same θ) in which they may be occupied. Not only do the impurities exclude gas molecules from certain sites, they also interact with gas molecules on nearby sites. As the impurities are fixed in position, we treat this additional interaction as an "external" potential acting on the gas molecules. There is such an effect also in a "regular" system near the boundaries.

Carrying out explicitly the transformation from the spin variables $S_i^z = \pm\frac{1}{2}$ to the occupation number variable $\rho_1 = S_i^z + \frac{1}{2} = (0, 1)$, we find, using (2.2),

$$\beta F(\theta) = \beta \sum_{i\in\theta} (\mu H + \tfrac{1}{4}\alpha_i) - \ln \Xi(\theta, z), \tag{5.1}$$

where $\Xi(\theta, z)$ is the grand partition function of a lattice gas, with fugacity $z = e^{2\beta\mu H}$, whose particles are confined to sites in θ and have a pair interaction potential between particles located at sites i and j:

$$\varphi_{ij} = \begin{cases} \infty, & i = j, \\ -4J_{ij}, & i \neq j, \end{cases} \tag{5.2}$$

and an external potential for a particle at site i equal to

$$-\alpha_i = 2\sum_{j\in\theta} J_{ij} = 2\sum_{j\in\Omega} J_{ij} - 2\sum_{\substack{j\notin\theta \\ j\in\Omega}} J_{ij}. \tag{5.3}$$

The first sum which is constant (except near the boundaries) amounts to a change in the fugacity while the second term corresponds formally to an external potential due to the impurities.

Continuing our transcription from spin system to lattice gas, we have from (3.1) and (3.2)

$$f_\Omega = p\mu H + p^2 \frac{1}{2N_\Omega} \sum_{j\in\Omega} \sum_{i\in\Omega} J_{ij} - \Pi_\Omega(p, z), \tag{5.4}$$

where

$$\Pi_\Omega(p, z) = \langle\langle \Pi(\theta, z)\rangle\rangle \equiv \beta^{-1} \frac{\langle\langle \ln \Xi(\theta, z)\rangle\rangle}{N_\Omega}, \tag{5.5}$$

with $\Pi(\theta, z)$ the "pressure" of the lattice gas confined to θ. The existence of $f = \lim_{N_\Omega\to\infty} f_\Omega$ then implies the existence of $\Pi(p, z) = \lim_{\Omega\to\infty} \Pi_\Omega$ for $0 \leq z \leq \infty$, $0 \leq p \leq 1$. [We have suppressed here the dependence on β, $0 \leq \beta \leq \infty$. The second term on the right side of (5.4) possesses a well defined limit as $N_\Omega \to \infty$ provided the J_{ij} satisfy (3.10).]

We shall now show, using methods developed for continuum systems and regular lattice systems,[15] that $\Pi(p, z)$ is analytic in z and p for $|z| < R(\beta)$ and $|p| \leq 1$ in the case where the interactions have a finite range. [By the symmetry $H \to -H$, this is also true $|z| > R^{-1}(\beta)$.] For purely ferromagnetic interactions $J_{ij} \geq 0$, the Lee–Yang theorem[16] guarantees analyticity in $|z|$ for $|z| < 1$, $H \neq 0$, and our results can be extended via a theorem due to Ruelle[17] to give analyticity for real positive p, $0 \leq p \leq 1$, when $H \neq 0$. (Results on analyticity in β for "regular" systems, which can also be extended to random systems, will be discussed elsewhere.[18])

We begin with the Mayer series

$$\beta\Pi(\theta, z) = \frac{1}{N_\Omega} \sum_{i=1}^{N_\Omega} \ln\left(\frac{1 - z}{z_i(\theta)}\right) = \sum_{l=1}^{\infty} b_l'(\theta) z^l, \tag{5.6}$$

where the $z_i(\theta)$ are the zeros of the grand partition function $\Xi(\theta, z)$ and

$$b_l'(\theta) = \frac{1}{N_\Omega l!} \sum_{\mathbf{x}_1\in\theta} \cdots \sum_{\mathbf{x}_l\in\theta} u_l(\mathbf{x}_1, \cdots, \mathbf{x}_l) \left[\prod_{i=1}^{l} e^{\beta\alpha(\mathbf{x}_i)}\right], \tag{5.7}$$

where $\mathbf{x}_i$ is the lattice vector of the ith site and the sum is over all lattice sites in θ, $\alpha(\mathbf{x}_i) = \alpha_i$, and u_l is the usual Mayer cluster function, $u_2(\mathbf{x}_1, \mathbf{x}_2) = \exp[-\beta\varphi(\mathbf{x}_1 - \mathbf{x}_2)] - 1$, etc. Taking the average, with $P(\theta)$, of $\Pi(\theta, z)$, we obtain

$$\beta\Pi_\Omega(p, z) = \sum_{l=1}^{\infty} b_l'(p, \Omega) z^l, \tag{5.8}$$

with

$$b_l'(p, \Omega) = \sum_\theta P(\theta) b_l'(\theta). \tag{5.9}$$

The averaging in (5.9) is facilitated by the fact that the probabilities of occupancy by impurities of different sites are independent and will be illustrated now for the case $l = 2$. Writing

$$b_2'(\theta) = \frac{1}{2N_\Omega} \sum_{i\in\theta} u_2(\mathbf{x}_i, \mathbf{x}_i) \prod_{j\in\theta} e^{4\beta J_{ij}} + \frac{1}{2N_\Omega} \sum_{i\neq j, i, j\in\theta} u_2(\mathbf{x}_i, \mathbf{x}_j) e^{4\beta J_{ij}} \prod_{\substack{k\in\theta \\ k\neq i,j}} e^{2\beta[J_{ik} + J_{jk}]}, \tag{5.10}$$

[15] See, for example, J. Groeneveld, *Proceedings of the I.U.P.A.P. Meeting, Copenhagen, 1966*, T. A. Bak, Ed. (W. A. Benjamin, Inc., N.Y., 1967), p. 110.

[16] T. D. Lee and C. N. Yang, Phys. Rev. **87**, 410 (1952).

[17] D. Ruelle (private communication).

[18] J. L. Lebowitz and O. Penrose (to be published).

we have

$$b_2'(p,\Omega)=\frac{1}{2N_\Omega}p\sum_{i\in\Omega}u_2(\mathbf{x}_i,\mathbf{x}_i)\prod_{j\in\Omega}[pe^{4\beta J_{ij}}+(1-p)]$$
$$+\frac{1}{2N_\Omega}p^2\sum_{\substack{i\neq j\\ i,j\in\Omega}}u_2(\mathbf{x}_i,\mathbf{x}_j)e^{4\beta J_{ij}}$$
$$\times\prod_{\substack{k\in\Omega\\ k\neq i,j}}[pe^{2\beta(J_{ik}+J_{jk})}+(1-p)]. \qquad (5.11)$$

Restricting ourselves to finite-range interactions, we see that $b_2'(p,\Omega)\to b_2'(p)$ as $N_\Omega\to\infty$:

$$b_2'(p)=\tfrac{1}{2}pu_2(\mathbf{x}_1,\mathbf{x}_1)\prod_j[1+p(e^{4\beta J_{1j}}-1)]$$
$$\cdot\ \tfrac{1}{2}p^2\sum_{j=1}u_2(\mathbf{x}_1-\mathbf{x}_j)e^{4\beta J_{ij}}$$
$$\times\prod_{k\neq 1,j}[1+p(e^{2\beta(J_{1k}+J_{jk})}-1)], \qquad (5.12)$$

the indices now running over the infinite lattice. Since J_{ij} has a finite range, only a finite number of the factors in the products in (5.12) are different from unity and thus $b_2'(p)$ is a polynomial. The same holds for every $b_l'(p)$. Further, for $|p|\leq 1$,

$$|b_2'(p,\Omega)|\leq\frac{1}{2N_\Omega}\sum_{i\in\Omega}|u_2(\mathbf{x}_i,\mathbf{x}_i)|\prod_{j\in\Omega}[1+|e^{4\beta J_{ij}}-1|]$$
$$+\frac{1}{2N_\Omega}\sum_{\substack{i\neq j\\ i,j\in\Omega}}|u_2(\mathbf{x}_i,\mathbf{x}_j)|$$
$$\times\prod_{\substack{k\in\Omega\\ k\neq i,j}}[1+|e^{2\beta(J_{ik}+J_{jk})}-1|]e^{4\beta J_{ij}}$$
$$\leq\frac{1}{2N_\Omega}\sum_{i,j\in\Omega}|u_2(\mathbf{x}_i,\mathbf{x}_j)|\,e^{2\beta\sum_k(|J_{ik}|+|J_{jk}|)}$$
$$\leq\tfrac{1}{2}\sum_j|u_2(\mathbf{x}_1-\mathbf{x}_j)|e^{2\beta\Phi}, \qquad (5.13)$$

where

$$\Phi=2\sum_k|J_{1k}|<\infty, \qquad (5.14)$$

and the sum over j and k goes over the infinite lattices. Similarly,

$$|b_l'(p,\Omega)|\leq\frac{1}{l!}\sum_{\{\mathbf{x}_2,\cdots,\mathbf{x}_l\}}|u_l(\mathbf{x}_1,\mathbf{x}_2,\cdots,\mathbf{x}_l)|\,e^{l\beta\Phi}$$
$$\leq\frac{1}{l!}e^{2l\beta\Phi}l^{l-2}B^{l-1}, \qquad (5.15)$$

where

$$B=\sum_j|u_2(\mathbf{x}_1-\mathbf{x}_j)|. \qquad (5.16)$$

The last inequality in (5.15) is due to Penrose.[19] It follows then from (5.15) that the power series (5.8), as well as the series

$$\beta\Pi(p,z)=\sum_{l=1}^{\infty}b_l'(p)z^l, \qquad (5.17)$$

[19] O. Penrose, *Statistical Mechanics, Foundations and Applications*, T. A. Bak, Ed. (W. A. Benjamin, Inc., N.Y., 1967), p. 101.

converge for $|p|\leq 1$:

$$|z|<R(\beta)=(Be^{2\beta\Phi+1})^{-1}. \qquad (5.18)$$

The convergence of (5.17) for $|z|<R$, $|p|\leq 1$ and the fact that the $b_l'(p)$ are analytic (polynomials) in p for finite-range potentials implies that $\Pi(p,z)$ and therefore f is analytic also in p for $|p|\leq 1$ and $|z|<R$.

For the case of purely ferromagnetic interactions, $J_{ij}\geq 0$ for all i,j, the Lee–Yang theorem[16] states that the roots of the grand partition function $\Xi(\theta,z)$, all lie on the unit circle $|z_i(\theta)|=1$, from which it follows that $\Pi(p,z)$ is analytic in z for $|z|<1$ and p real positive $0\leq p\leq 1$. Hence there exists some K such that

$$|b_l'(p)|<K, \qquad \text{for} \quad 0\leq p\leq 1, \qquad (5.19)$$

and

$$|b_l'(p)|<K/R^l, \quad \text{for} \quad |p|\leq 1, \qquad (5.20)$$

where (5.20) follows from our previous results. Combining (5.19), (5.20), and the fact that $b_l'(p)$ is analytic in p, it is possible to show that $\Pi(p,z)$ are analytic in p along the real axis $0\leq p\leq 1$ for all $|z|<1$, for systems where $J_{ij}>0$.

6. ADDITIONAL RESULTS

In addition to the results established in this paper, there are many questions relating to random spin systems which we have not touched on at all. These concern the dependence of the correlation functions, magnetization, including spontaneous magnetization, and the critical point indices on p. There are several results which can be established easily for random Ising spin systems, which we shall state here without proof.

(1) The existence and analyticity of the correlation functions for $|z|\leq R$, $|p|\leq 1$.

For systems with purely ferromagnetic interactions we also have:

(2) The average magnetization per spin is a monotonically increasing function of p (as well as of β and H).

(3) It follows from (2) that the critical temperature $T_c(p)$ (onset of spontaneous magnetization) is also a monotonic function of p.

(4) If T_0 is the critical temperature obtained from mean field theory for a "regular" system, then $T_c(p)\leq pT_0$.

(5) For nearest-neighbor interactions, the concentration p_0 at which spontaneous magnetization occurs at $T=0$ is greater than or equal to the critical percolation concentration p_0.

APPENDIX: INEQUALITY USED IN EQUATION (4.12)

Let z be a random variable which takes two possible values W and 1, with probability p and $q = 1 - p$, respectively. We assume that W is nonnegative. For $\alpha \geq 1$,

$$f_\alpha(z) = z^\alpha \tag{A1}$$

is a convex function for positive z; hence[20]

$$(W^\alpha p) + q = \langle f_\alpha(z)\rangle \geq f_\alpha(\langle z\rangle) = (Wp + q)^\alpha. \tag{A2}$$

Thus, for any $\gamma > 0$,

$$(W^\alpha p + q)^\gamma \geq (Wp + q)^{\alpha\gamma}. \tag{A3}$$

We use (A3) twice in obtaining the second inequality in (4.12): first with $\alpha = 3$, $\gamma = k_1/3$, and $W = y$; next with $\alpha = \frac{3}{2}$, $\gamma = 2k_2/3$, and $W = y^2$.

[20] G. H. Hardy, J. E. Littlewood, and G. Polya, *Inequalities* (Cambridge University Press, Cambridge, England, 1964), Chap. III.

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Two-Center Coulomb and Hybrid Integrals*

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(Received 9 August 1967)

Coulomb and hybrid integrals are shown to be related to overlap integrals. They are expressed by an integral whose integrand is an overlap integral plus a finite sum of overlap integrals.

INTRODUCTION

Three general numerical methods for the calculation of two-center Coulomb and hybrid integrals are: (1) the application of the exchange integral method based on the Neumann expansion,[1] (2) the numerical integrations on the second electron after the integration over the first electron has been carried out analytically,[2] and (3) the expression for Coulomb integrals based on the Fourier convolution theorem.[3]

For the Coulomb integrals a general analytical method has recently been developed by O-Ohata and Ruedenberg. An essential feature of their approach is the reduction of the Coulomb integrals to an integration over overlap integrals.[4]

In the present note, it is shown that there exists yet another way of expressing the Coulomb and hybrid integrals by integration over overlap integrals.

* Work was performed in the Ames Laboratory of the Atomic Energy Commission, Contribution No. 1959.

† Present address: Department of Chemistry, Rensselaer Polytechnic Institute, Troy, New York.

[1] K. Ruedenberg, *Molecular Orbitals in Chemistry, Physics, and Biology*, B. Pullman and P. O. Löwdin, Eds. (Academic Press Inc., New York, 1964), p. 215.

[2] A. C. Wahl, P. E. Cade, and C. C. J. Roothaan, J. Chem. Phys. **41**, 2578 (1964); see also R. Christoffersen and K. Ruedenberg, J. Chem. Phys. **47**, 1855 (1967).

[3] M. Geller, J. Chem. Phys. **41**, 4006 (1964).

[4] K. O-Ohata and K. Ruedenberg, J. Math. Phys. **7**, 547 (1966).

ATOMIC ORBITALS AND CHARGE DISTRIBUTIONS

A normalized Slater-type atomic orbital on center A for electron i is given by

$$(Anlm, \zeta, i) = N_n(2\zeta)^{n+\frac{1}{2}} r_{Ai}^{n-1} \times \exp(-\zeta r_{Ai}) Y_{lm}(\theta_{Ai}, \phi_{Ai}), \tag{1}$$

$$N_n = (2n!)^{-\frac{1}{2}}, \tag{1'}$$

where Y_{lm} may be either a real or complex spherical harmonic.[5] While overlap integrals usually occur between such atomic orbitals, two-center Coulomb integrals are commonly defined between certain standard "charge distributions"

$$[Anlm, \zeta, i] = (M_{nl}/N_n)(Anlm, \zeta, i), \tag{2}$$

which differ from the orbitals of Eq. (1) by certain constants. Since several conventions have been employed for the factors M_{nl}, its specific form will not be used in the sequel.[6] The electron index i may be omitted for convenience.

[5] The real spherical harmonics used are those defined by C. C. J. Roothaan, J. Chem. Phys. **19**, 1445 (1951).

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$$M_{nl} = [(2l + 1)/\pi]^{\frac{1}{2}} \zeta_c^{\frac{3}{2}}/[(n + l + 1)!\, 2^{n-l-1}],$$

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Also, ζ_c is replaced by 2ζ in both references.

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Also, ζ_c is replaced by $2\bar{\zeta}$ in both references.

ONE-ELECTRON POTENTIALS

The one-electron potential arising from the charge distribution (2) is defined as

$$\langle Anlm; \zeta_A, 2\rangle = \int dV_1 [Anlm; \zeta_A, 1]^*/r_{12}. \quad (3)$$

Use of the Laplace expansion on center A for r_{12}^{-1} yields[7]

$$\langle Anlm\rangle = [\pi/(2l+1)]2^{n+5/2}\zeta_A^{-\frac{1}{2}} M_{nl} Y_{lm}^*(\theta_{A2}, \phi_{A2}) \times \left[(n+l+1)\int_0^1 dt\, t^{n+l} s^n e^{-st} + (n-l)!\, e^{-s} \sum_{k=0}^{n-l-1} s^{k+l}/k!\right], \quad (4)$$

where

$$t = r_{A1}/r_{A2}, \quad s = \zeta_A r_{A2}.$$

It is possible to express the potential $\langle Anlm\rangle$ in terms of the atomic orbitals of Eq. (1). Two expressions, both having the form

$$\langle Anlm\rangle = M_{nl}[2^{n+1-l}\zeta_A^{-2}\pi/(2l+1)] \times \left\{a(nl)\int_0^1 dt\, t^{l-\frac{3}{2}}(A(\lambda+1)lm; t\zeta_A)^* + \sum_{k=0}^{n-l-1} b_k(nl)(A(l+k+1)lm; \zeta_A)^*\right\}, \quad (5)$$

are possible. The first expression is essentially identical with Eq. (4), and is given by the coefficients

$$a(nl) = a'(nl) = (n+l+1)/(2^{n-l}N_{n+1}), \quad (6a)$$

$$b_k(nl) = b_k'(nl) = (n-l)!/(2^k k!\, N_{l+k+1}), \quad (6b)$$

and

$$\lambda = n. \quad (6c)$$

The second expression is obtained with the help of the identity

$$s^n \int_0^1 dt\, t^{n+l} e^{-st} = (n+l)! \left\{s^l/(2l)! \int_0^1 dt\, t^{2l} e^{-st} - e^{-s} \sum_{k=0}^{n-l-1} s^{k+l}/(k+2l+1)!\right\}, \quad (7)$$

and it is given by the coefficients

$$a(nl) = a''(nl) = (n+l+1)!/[(2l)!\, N_{l+1}], \quad (8a)$$

$$b_k(nl) = b_k''(nl) = (n+l+1)!\, c_k^{nl}/(2^k N_{l+k+1}), \quad (8b)$$

and

$$\lambda = l, \quad (8c)$$

where the coefficients

$$c_k^{nl} = (n-l)!/[(n+\quad+1)!\, k!] - [(k+2l+1)!]^{-1} \leq 0 \quad (9)$$

can be calculated from the recurrence scheme

$$c_{n-l}^{nl} = 0,$$
$$c_{k-1}^{nl} = kc_k^{nl} - (2l+1)/(2l+k+1)!, \quad (9')$$
$$k = 1, 2, \cdots (n-l).$$

TWO-CENTER COULOMB INTEGRALS

The two-center Coulomb integral between two basic charge distributions is defined by

$$\int dV_1 \int dV_2 [Anlm; \zeta_A, 1]^*[Bn'l'm'; \zeta_B, 2]/r_{12} = \delta_{mm'} C_{nn'}^{ll'm}(R, \zeta_A, \zeta_B), \quad (10)$$

where R is the distance $\overline{AB}$. Using the results of the preceding section, one finds

$$C_{nn'}^{ll'm} = (M_{n'l'}/N_{n'}) \int dV \langle Anlm; \zeta_A\rangle(Bn'l'm'; \zeta_B) \quad (11)$$

or

$$C_{nn'}^{ll'm} = M_{nl}M_{n'l'}[2^{n+1-l}\zeta_A^{-2}N_{n'}^{-1}\pi/(2l+1)] \times \left\{a(nl)\int_0^1 dt t^{l-\frac{3}{2}} S_{\lambda+1,n'}^{ll'm}(t\zeta_A, \zeta_B) + \sum_{k=0}^{n-l-1} b_k(nl) S_{l+k+1,n'}^{ll'm}(\zeta_A, \zeta_B)\right\}, \quad (12)$$

where

$$S_{nn'}^{ll'm}(\zeta_A, \zeta_B) = \int dV (Anlm; \zeta_A)^*(Bnlm; \zeta_B) \quad (13)$$

is the two-center overlap integral between Slater-type atomic orbitals. Equation (12) gives rise to two different formulas for the Coulomb integral depending upon which of the two sets of coefficients, viz., Eqs. (6a)–(6c) or Eqs. (8a)–(8c), are used. In both cases the Coulomb integral is reduced to a one-dimensional integration over overlap integrals.

HYBRID INTEGRALS

A two-center hybrid integral can be defined by

$$H_{n\ n'\ n''}^{lml'm'l''m''} = \int dV_1 \int dV_2 [Anlm; \zeta_A, 1]^* r_{12}^{-1} \times (An'l'm'; \zeta_A', 2)(Bn''l''m''; \zeta_B, 2) \quad (14)$$

or

$$H_{n\ n'\ n''}^{lml'm'l''m''} = \int dV \langle A(nlm)\rangle(An'l'm'; \zeta_A')(Bn''l''m''; \zeta_B). \quad (15)$$

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[7] The Laplace expansion is given, e.g., in Eyring, Walter, and Kimball, *Quantum Chemistry* (John Wiley & Sons, Inc., New York, 1957), p. 371. The term in brackets is obtained after integrating by parts once on

$$\int_0^1 dt\, t^{n+l+1}s^{n+1}e^{-st} + (n-l)!\, e^{-s}\sum_{k=0}^{n-l} s^{k+l}/k!,$$

which follows directly from Eq. (3).

TABLE I. Sign of m_1m_2 and (m_1+m_2).

(m_1m_2)	(m_1+m_2)	E_+	E_-
+	+	1	1
+	−	−1	1
−	+	1	−1
−	−	1	1
−	0	1	0
0	+, 0, −	1	0

For complex orbitals, it vanishes unless $m'' = m - m'$, and for real orbitals it vanishes unless $m'' = M+$ or $M-$, where

$$M\pm = \operatorname{sign}(m)\operatorname{sign}(m')\,|\{|m| \pm |m'|\}|$$

and

$$\operatorname{sign}(X) = X/|X|.$$

(For the real spherical harmonics, $m > 0$ denotes a $\cos|m|\,\varphi$ dependence and $m < 0$ denotes a $\sin|m|\,\varphi$ dependence.) When Eq. (5) is substituted in Eq. (15), there appear products of two orbitals on the same center A. For these products, the following expression can be introduced:

$$\begin{aligned}(Aplm;\zeta_A)^*(Aql'm';\zeta_A') &= 2^{\frac{3}{2}}\zeta_A^{p+\frac{1}{2}}\zeta_A'^{q+\frac{1}{2}}N_pN_q[(\zeta_A+\zeta_A')^{p+q-\frac{1}{2}}N_{p+q-1}]^{-1}\\ &\quad\times\sum_{LM}C_{LM}^{lml'm'}(A(p+q-1)LM;\zeta_A+\zeta_A')^*, \quad (16)\end{aligned}$$

where the summation over L is restricted by

$$|l-l'| \le L \le (l+l'), \quad l+l'+L = \text{even}. \quad (16')$$

For complex orbitals the coefficients $C_{LM}^{l'm'm'}$ vanish except for $M = m - m'$, in which case they are[8]

$$C_{LM}^{lml'm'} = (-1)^m C(l(-m), l'm', LM), \quad (17)$$

where

$$\begin{aligned}C(lm, l'm', LM) &= \delta_{m+m'+M,0}[(2l+1)(2l'+1)(2L+1)/4\pi]^{\frac{1}{2}}\\ &\quad\times\begin{pmatrix} l & l' & L\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} l & l' & L\\ m & m' & M\end{pmatrix}. \quad (18)\end{aligned}$$

For real orbitals the coefficients vanish unless $M = M+$ or $M-$, in which case they are

$$\begin{aligned}C_{LM_+}^{lml'm'} &= E_+(-1)^{m_1+m_2}[1+\delta_{0,m_1m_2}]^{\frac{1}{2}}\\ &\quad\times C(l(-|m|), l'(-|m'|), L(|m_1|+|m_2|)) \quad (19)\end{aligned}$$

and

$$\begin{aligned}C_{LM_-}^{lml'm'} &= E_-(-1)^{\max(|m_1|,|m_2|)}[1+\delta_{0,m_1-m_2}]^{\frac{1}{2}}\\ &\quad\times C(l(-|m|), l'|m'|, L(|m_1|-|m_2|)), \quad (19')\end{aligned}$$

[8] The $3j$ symbol $\begin{pmatrix} l_1 & l_2 & L\\ m_1 & m_2 & M\end{pmatrix}$ is given, e.g., in A. E. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, N.J., 1957), p. 46, Eq. 3.7.3.

where E_+ and E_- depend on the sign of m_1m_2 and (m_1+m_2) as given in Table I.

After substituting Eqs. (5) and (16) into Eq. (15), one obtains the result for the hybrid integrals:

$$\begin{aligned}H_{n\;n'\;n''}^{lml'm'l''m''} &= M_{nl}[2^{n-l+1}\zeta_A^{-2}\pi/(2l+1)]2^{\frac{3}{2}}N_{n'}\zeta_A'^{n'+\frac{1}{2}}\sum_L C_{Lm''}^{lm\;\;l'm'}\\ &\quad\times\Big\{(a(nl)N_{\lambda+1}/N_{\lambda+n'})\int_0^1 dt\,t^{l-\frac{3}{2}}\\ &\quad\times[(t\zeta_A)^{\lambda+\frac{3}{2}}/(t\zeta_A+\zeta_A')^{\lambda+n'+\frac{1}{2}}]\,S_{\lambda+n'n''}^{L\;\;l''m''}(t\zeta_A+\zeta_A',\zeta_B)\\ &\quad+\sum_{k=0}^{n-l-1} b_k(nl)N_{l+k+1}\zeta_A^{l+k+\frac{3}{2}}[N_{n'+l+k}(\zeta_A+\zeta_A')^{n'+l+k+\frac{1}{2}}]^{-1}\\ &\quad\times S_{n'+l+kn''}^{L\;\;l''m''}(\zeta_A+\zeta_A',\zeta_B)\Big\}. \quad (20)\end{aligned}$$

Depending upon the choice of $a(nl)$, $b_k(nl)$, and λ, one has again two possible representations.

Both real and complex orbitals yield the same form, since the spherical harmonic $Y_{l''m''}(\theta,\phi)$ is orthogonal to all but (at most) one harmonic in the expansion of the product of Slater-type atomic orbitals encountered in the modified potential.

DISCUSSION

The two-center Coulomb and hybrid integrals are both obtained by a numerical integration and summation over overlap integrals. The first of the two expressions given for these integrals will yield greater accuracy, since the sign of the terms will be that of the overlap integral involved; namely, for s-type integrals, only positive contributions occur. The second expression has the advantage that the principal quantum numbers needed are smaller for the overlap integrals in the integrand. Therefore, if the method developed by Ruedenberg *et al.*[9] is used, a smaller table can be generated, thus saving computation time; however, to obtain these Coulomb and hybrid integrals, one can expect a difference between two large numbers and a corresponding loss in accuracy.

Coulomb integrals have been calculated through $3d\delta$ cases for which a 16-point Gauss–Legendre numerical integration gave 7-figure accuracy.

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[9] K. Ruedenberg, K. O-Ohata, and D. G. Wilson, J. Math. Phys. 7, 539 (1966).

Quasibinomial Representations of Clebsch–Gordan Coefficients. I. Square Symbol*

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Quasibinomial representation of Clebsch–Gordan coefficients is symbolized by a Regge-like square. Rules for exchange of rows and columns of this square are derived. Thus our various quasibinomial forms can be read off from this square directly.

1. INTRODUCTION

In previous papers[1,2] we have described an operator method for the addition of two angular momenta, according to which one of the angular momenta is added to another in n steps of half each, the symmetrization of the n half angular momenta, so introduced, being taken into account at each step explicitly. It was shown that the determination of the angular-momentum eigenfunctions at each step can be performed easily by employing a particular representation of Löwdin's projection operators.[3] The use of these n operators was seen to be responsible for the appearance of products of n parameters in the general expression of the Clebsch–Gordan coefficients (CGc). Consequently, the notation of generalized power[4] was utilized to write our basic expression of CGc[5] in a clear and concise form. It was found that various *quasibinomial* (q.b.) representations of this expression of CGc can be derived[6] when the algebra of generalized power (g.p.) was taken seriously. These various q.b. forms are actually nothing but the different symmetry relations between the symmetrical formulas of CGc. An extension of the formalism to include Wigner-type unsymmetrical formulas with the aid of negative g.p. is possible. The negative q.b. representations of CGc, so obtained, can be treated exactly as their "positive" counterpart.[7]

In the days of preoccupation with the graphical and symbolic methods,[8] it is not inexpedient to look for symbolization of our q.b. representations. To that end, we employ Regge-like squares,[9,10] from which one is able to read off the corresponding q.b. representations just by inspection. Naturally, these squares are related to one another as the various q.b. forms themselves. In the following, we give the proofs of the rules by means of which any one square can be transformed into any other. We illustrate the rules by an example. Finally, the difference between our CGc square and that of Regge is examined.

2. RECAPITULATION

A general square symbol for CGc is defined by[1,6]

$$\boxed{\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix}} = \mathcal{N}' \frac{(\alpha'!)^{\frac{1}{2}}}{\alpha!} \left[\frac{\alpha''^{(\alpha''-\gamma')}}{\beta'^{(\alpha''-\gamma')}} \beta^{(\alpha''-\beta')} \gamma^{(\alpha''-\gamma')} \right]^{\frac{1}{2}} \times ((\beta'\gamma'' - \beta''\gamma'))^{(\alpha)}, \quad (1)$$

where $\mathcal{N}'$ is a constant factor which remains the same on permutation of rows or columns. It is given by Eq. (15″). Here we employ the notation of generalized power defined in general as[11]

$$x^{(n)} \equiv x(x-1)\cdots(x-n+1) = \binom{x}{n} n!, \quad x \geq n. \quad (2)$$

The double parentheses in (1) symbolize our q.b. expansion, a general one being defined as

$$((ax - by))^{(n)} = \sum_\tau (-1)^\tau \binom{n}{\tau} a^{(n-\tau)} x^{(n-\tau)} b^{(\tau)} y^{(\tau)}. \quad (3)$$

Note

$$((ax - by))^{(n)} = (-1)^n ((by - ax))^{(n)}, \quad (4)$$

and also the product formula

$$x^{(n)} = x^{(m)} (x - m)^{(n-m)}. \quad (5)$$

* Based on a part of the author's dissertation, University of Tübingen, 1966 (unpublished).

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[1] S. M. R. Ansari, Nuovo Cimento 38, 1883 (1965). In this note, we have reported only the essential results of this series of our papers (see below).

[2] S. M. R. Ansari, Fortschr. Physik 15, 707 (1967).

[3] P.-O. Löwdin, Rev. Mod. Phys. 36, 966 (1964).

[4] See definition (2) above.

[5] See, for example, Ref. 1, expression (3), or Ref. 2, expression (3.35).

[6] S. M. R. Ansari, Fortschr. Physik 15, 729 (1967). For the definition of q.b. expansion see Eq. (3) above.

[7] S. M. R. Ansari, J. Math. Phys. 9, 1299 (1968) (following paper).

[8] J.-N. Massot, E. El-Baz, and J. Lafoucziere, Rev. Mod. Phys. 39, 288 (1967); G. L. Strobel, Nucl. Phys. A96, 229 (1967); G. Flach, Nucl. Phys. 69, 68 (1965).

[9] Our square is different from the Regge square, Ref. 10. We examine this point in Sec. 5 in detail.

[10] T. Regge, Nuovo Cimento 10, 544 (1958).

[11] For a résumé of generalized powers (factorials) we recommend especially Ref. 6, Sec. 2. See also C. Jordan, *Calculus of Finite Differences* (Chelsea Publishing Co., New York., 1950).

Since (3) contains only the positive g.p.,[12] we may call it also a positive q.b. expansion and, consequently, the square symbolizing it a *positive* CGc square.

We recall the relations between the various elements of the square, viz.,[13] *the sum of elements in a row (or column) is always an integer, and all such sums are equal,* i.e.,

$$\alpha + \alpha' + \alpha'' = \beta + \beta' + \beta'' = \alpha'' + \beta'' + \gamma'' = \text{etc.} \tag{6}$$

3. PROOFS OF THE RULES

Rule I: The square is invariant with respect to transposition.

By Eq. (1), the transposed square is given by

$$\boxed{\begin{matrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{matrix}} = \mathcal{N}' \frac{(\beta!)^{\frac{1}{2}}}{\alpha!} \left[\frac{\gamma^{(\gamma-\beta'')}}{\beta'^{(\gamma-\beta'')}} \alpha'^{(\gamma-\beta')} \alpha''^{(\gamma-\beta'')} \right]^{\frac{1}{2}} \times ((\beta'\gamma'' - \gamma'\beta''))^{(\alpha)}. \tag{7}$$

Since by (6) $\alpha'' + \beta'' - \gamma - \gamma' = 0$, using (5), one gets

$$\gamma^{(\gamma-\beta'')} = \gamma^{(\gamma-\beta'')} \beta''^{(\alpha''+\beta''-\gamma-\gamma')} = \gamma^{(\alpha''-\gamma')}$$

and

$$\alpha''^{(\gamma-\beta'')} = \alpha''^{(\gamma-\beta'')} (\alpha'' + \beta'' - \gamma)^{(\alpha''+\beta''-\gamma-\gamma')} = \alpha''^{(\alpha''-\gamma')}.$$

Similarly,

$$\beta'^{(\gamma-\beta'')} = \beta'^{(\alpha''-\gamma')}.$$

Again by (5) and (6)

$$\beta!\, \alpha'^{(\gamma-\beta')} = \beta!\, \alpha'^{(\alpha'-\beta)} \beta^{(\gamma+\beta-\alpha'-\beta')} = \alpha'!\, \beta^{(\alpha''-\beta')}.$$

Substituting these relations in (7), one sees immediately that

$$\boxed{\begin{matrix} \alpha & \alpha' & \alpha'' \\ \beta & \beta' & \beta'' \\ \gamma & \gamma' & \gamma'' \end{matrix}} = \boxed{\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix}}.$$

Rule II: A square obtained by exchanging the second and the third row (column) is always multiplied by $(-1)^\alpha$, α being the first element.

Now the new square is given as follows:

$$\boxed{\begin{matrix} \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \\ \alpha' & \beta' & \gamma' \end{matrix}} = \mathcal{N}' \frac{(\alpha''!)^{\frac{1}{2}}}{\alpha!} \left[\frac{\alpha'^{(\alpha'-\gamma'')}}{\beta''^{(\alpha'-\gamma'')}} \beta^{(\alpha'-\beta'')} \gamma^{(\alpha'-\gamma'')} \right]^{\frac{1}{2}} \times ((\beta''\gamma' - \beta'\gamma''))^{(\alpha)}. \tag{8}$$

First, by virtue of (4)

$$((\beta''\gamma' - \beta'\gamma''))^{(\alpha)} = (-1)^\alpha ((\beta'\gamma'' - \beta''\gamma'))^{(\alpha)}. \tag{9}$$

Second, we utilize the following factorization due to (5) and (6):

$$\begin{aligned} \beta^{(\alpha'-\beta'')} &= \beta^{(\alpha''-\beta')} (\beta + \beta' - \alpha'')^{(\alpha'+\beta'-\alpha''-\beta'')} \\ &= \beta^{(\alpha''-\beta')} \gamma''^{(\gamma''-\gamma')}. \end{aligned}$$

Similarly,

$$\gamma^{(\alpha'-\gamma'')} = \gamma^{(\alpha''-\gamma')} \beta''^{(\beta''-\beta')}.$$

With these, the square-root factor in (8) becomes

$$\begin{aligned} &[\alpha''!\, \gamma''^{(\gamma''-\gamma')} \alpha'^{(\alpha'-\gamma'')}] \frac{\beta''^{(\beta''-\beta')}}{\beta''^{(\alpha'-\gamma'')}} [\beta^{(\alpha''-\beta')} \gamma^{(\alpha''-\gamma')}] \\ &\quad = [\alpha'!\, \alpha''^{(\alpha''-\gamma')}][\beta'^{(\alpha''-\gamma')}]^{-1}[\beta^{(\alpha''-\beta')} \gamma^{(\alpha''-\gamma')}], \end{aligned} \tag{9'}$$

which is exactly the square-root factor appearing in (1). Here, we have used the following relations due to (5) and (6):

$$\alpha'^{(\alpha'-\gamma'')} = \alpha'^{(\alpha'-\alpha'')} \alpha''^{(\alpha''-\gamma')} \gamma'^{(\gamma'-\gamma'')},$$

$$\gamma''^{(\gamma''-\gamma')} \gamma'^{(\gamma'-\gamma'')} = 1,$$

and

$$\beta''^{(\alpha'-\gamma'')} = \beta''^{(\beta''-\beta')} \beta'^{(\alpha'+\beta'-\beta''-\gamma'')} = \beta''^{(\beta''-\beta')} \beta'^{(\alpha''-\gamma')}.$$

Thus with (9) and (9′), the rule

$$(-1)^\alpha \boxed{\begin{matrix} \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \\ \alpha' & \beta' & \gamma' \end{matrix}} = \boxed{\begin{matrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{matrix}}$$

is also established.

In the case of exchanging the second and third column, the validity of the rule can be seen immediately, if one takes note of the relationship

$$\frac{\alpha''^{(\alpha''-\beta')}}{\gamma'^{(\alpha''-\beta')}} = \frac{\alpha''^{(\alpha''-\gamma')} \gamma'^{(\gamma'-\beta')}}{\gamma'^{(\gamma'-\beta')} \beta'^{(\alpha''-\gamma')}} = \frac{\alpha''^{(\alpha''-\gamma')}}{\beta'^{(\alpha''-\gamma')}}.$$

Rule III: An exchange of the first and second row of a square gives rise to an additional factor $(-1)^{\beta''}$, β'' being the element in the third row and second column.

[12] In contradistinction to (2), one may define the negative g.p. by $(-x)^{(n)} = (-1)^n (x + n - 1)^{(n)}$, which plays an important role in the "negative" q.b. representations of the Wigner-type unsymmetrical formulas of CGc. See Ref. 7.

[13] See Ref. 6, Sec. 6. This property defines what is called a semimagic square in the theory of partitions; see P. A. Macmahon, *Combinatory Analysis* (Cambridge University Press, Cambridge, England, 1916), Vol. II, Chap. VII, p. 160 *et seq.* The symmetry of such a square has been studied on the basis of permutations by T. Shimpuku [Nuovo Cimento **27**, 874 (1963)].

That is,[14a]

$$(-1)^{\beta''}\boxed{\begin{matrix}\alpha' & \beta' & \gamma' \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma''\end{matrix}} = \boxed{\begin{matrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma''\end{matrix}}. \tag{10}$$

From (1), we have

$$\boxed{\begin{matrix}\alpha' & \beta' & \gamma' \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma''\end{matrix}} = \mathcal{N}'\frac{(\alpha!)^{\frac{1}{2}}}{\alpha'!}\left[\frac{\alpha''^{(\alpha''-\gamma)}}{\beta^{(\alpha''-\gamma)}}\beta'^{(\alpha''-\beta)}\gamma'^{(\alpha''-\gamma)}\right]^{\frac{1}{2}} \times ((\beta\gamma''-\beta''\gamma))^{(\alpha')}, \tag{11}$$

in which, by definition (3), the q.b. expansion is given by

$$((\beta\gamma''-\beta''\gamma))^{(\alpha')} = \sum_x (-1)^x \binom{\alpha'}{x}\beta''^{(x)}\gamma^{(x)}\beta^{(\alpha'-x)}\gamma''^{(\alpha'-x)}. \tag{12}$$

In order to prove (10), we have first of all to deduce from it the q.b. expansion with α as exponent. For that purpose we proceed as follows:

On substituting in (12) the relations

$$\beta^{(\alpha'-x)} = \beta^{(\alpha'-\beta'')}\gamma'^{(\beta''-x)} \quad \text{[by (5) and (6)]},$$

$$\gamma''^{(\alpha'-x)} = \gamma''^{(\alpha'-\beta'')}\alpha^{(\beta''-x)} \quad \text{[by (5) and (6)]},$$

and the "exchange"

$$\binom{\alpha'}{x}\beta''^{(x)} = \binom{\beta''}{x}\alpha^{(x)}, \qquad \text{[by (2)]}$$

we get, with $\beta'' - x = y$,

$$((\beta\gamma''-\beta''\gamma))^{(\alpha')} = (-1)^{\beta''}((\alpha'\gamma-\alpha\gamma'))^{(\beta'')}\beta^{(\alpha'-\beta'')}\gamma''^{(\alpha'-\beta'')}. \tag{12'}$$

We now make use of the exchange

$$\binom{\beta''}{y}\alpha^{(y)} = \binom{\alpha}{y}\beta''^{(y)}$$

in $((\ \ ))^{(\beta'')}$ above, and proceed exactly as before. We obtain finally the q.b. expansion with α as exponent, namely,

$$((\ \ ))^{(\alpha')} = (-1)^{\beta''}((\beta'\gamma''-\beta''\gamma'))^{(\alpha)} \times \alpha'^{(\beta''-\alpha)}\gamma^{(\beta''-\alpha)}\beta^{(\alpha'-\beta'')}\gamma''^{(\alpha'-\beta'')}. \tag{13}$$

In this way we have the factor $(-1)^{\beta''}$ also. The only thing we have to show now is that all the factors appearing before $((\ \ ))^{(\alpha)}$, obtained by putting (13) into (11), can be recast into the same form as in (1).

To this end, let us denote these factors by $\{f\}$ and make the following substitutions:

$$\gamma^{(\beta''-\alpha)} = \gamma^{(\alpha''-\gamma')}\beta''^{(\beta''-\beta')},$$

$$\beta^{(\alpha'-\beta'')} = \beta^{(\alpha''-\beta')}\gamma''^{(\gamma''-\gamma')},$$

and

$$\alpha''^{(\alpha''-\gamma)} = \alpha''^{(\alpha''-\gamma')}\gamma'^{(\gamma'-\gamma)}.$$

Then

$$\{f\} = \mathcal{N}'(\alpha''^{(\alpha''-\gamma')}\beta^{(\alpha''-\beta')}\gamma^{(\alpha''-\gamma')})^{\frac{1}{2}}\frac{(\alpha!)^{\frac{1}{2}}}{\alpha'!}\{\alpha'^{(\beta''-\alpha)}\gamma''^{(\alpha'-\beta'')}\}$$
$$\times [\gamma''^{(\gamma''-\gamma')}\gamma'^{(\gamma'-\gamma)}\gamma^{(\beta''-\alpha)}\beta'^{(\alpha''-\beta)}]^{\frac{1}{2}}$$
$$\times \{\beta^{(\alpha'-\beta'')}\gamma'^{(\alpha''-\gamma)}\beta''^{(\beta''-\beta')}/\beta^{(\alpha''-\gamma)}\}^{\frac{1}{2}}.$$

Again, with the aid of (5) and (6), one may simplify the brackets after a little algebra and obtain the following results:

$$\{\ \ \} = \alpha'^{(\alpha'-\alpha)}, \quad \gamma'' = \alpha' + \alpha - \beta'',$$

$$[\ \ ]^{\frac{1}{2}} = \gamma''^{(\gamma''-\beta')}\beta'^{(\alpha''-\beta)} = \gamma''^{(0)} = 1$$

$$(\text{as } \beta' = \alpha + \gamma - \beta'', \gamma'' + \alpha'' = \beta + \beta'),$$

and

$$\{\ \ \}^{\frac{1}{2}} = \beta^{(\beta-\beta')}/\beta^{(\alpha''-\gamma)} = \alpha'^{(\alpha'-\beta')}.$$

Note also that

$$\alpha'^{(\alpha'-\alpha)}(\alpha!\,\alpha'^{(\alpha'-\beta')})^{\frac{1}{2}} = [\alpha!(\beta'^{(\alpha''-\gamma')})^{\frac{1}{2}}]^{-1}$$
$$(\alpha = \beta' + \gamma' - \alpha'').$$

Consequently, $\{f\}$ goes into the factor which appears in (1).

A corresponding rule holds also for the first and second column and can be proved in the same way as above. It states:

Rule IV: Exchanging the first and second column in a square yields an additional factor $(-1)^{\gamma'}$, γ' being the element of the third column and second row.

That is,

$$\boxed{\begin{matrix}\alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma''\end{matrix}} = (-1)^{\gamma'}\boxed{\begin{matrix}\beta & \alpha & \gamma \\ \beta' & \alpha' & \gamma' \\ \beta'' & \alpha'' & \gamma''\end{matrix}}.$$

4. AN ILLUSTRATION

We start with our basic square,[14b] viz.,

$$\boxed{\begin{matrix}\mu & a_{n-\mu} & b_{\mu+1} \\ 2l_1 - \mu & z & s \\ n - \mu & b_{z+1} & a_z\end{matrix}}. \tag{14}$$

Note first the following relations between our parameters and the quantum numbers usually employed in the theory of addition of two angular

[14a] Compare this and the following Rule IV with those of Regge, Ref. 10.

[14b] Reference 6, Eq. (6.1).

momenta:

$$\mu = l_1 + l_2 - j, \quad a_{n-\mu} = j + M, \quad b_{\mu+1} = j - M,$$
$$2l_1 - \mu = l_1 - l_2 + j, \quad z = l_2 - m_2, \quad s = l_2 + m_2,$$
$$n - \mu = l_2 - l_1 + j, \quad b_{z+1} = l_1 - m_1, \quad a_z = l_1 + m_1. \tag{15}$$

The CGc matrix

$$C(l_1 l_2 j; m_1 m_2 M) = C_{jm_2} \tag{15'}$$

then in our notation is $C_{\mu z}$, or, for fixed μ, simply C_z, the q.b. representation of which is C (exponent). Thus, for instance, the q.b. representation associated with the square (14) is $C(\mu)$, which can be immediately written out by definition (1).[15] In terms of our parameters (15), the factor

$$\mathcal{N}' = [(2l_1 + n - 2\mu + 1)/(2l_1 + n - \mu + 1)!]^{\frac{1}{2}}. \tag{15''}$$

We may mention that $C(\mu)$, when expressed in terms of ordinary factorials, is the well-known Racah symmetrical formula for CGc.[16]

As an application of the rules derived above, we now show that[17]

$$C(\mu) = (-1)^{z+\mu} C(b_{\mu+1}). \tag{16}$$

We may recall that it is in fact a symmetry relation[18]:

$$\frac{1}{(2j+1)^{\frac{1}{2}}} C(l_1 l_2 j; m_1 m_2 M)$$
$$= \frac{(-1)^{l_1 - j + m_2}}{(2l_1 + 1)^{\frac{1}{2}}} C(j l_2 l_1; M, -m_2, m_1).$$

If we symbolize the operation of exchanging the rows or columns by arrows, we have

$$\boxed{\begin{array}{ccc} \mu & a_{n-\mu} & b_{\mu+1} \\ 2l_1 - \mu & z & s \\ n - \mu & b_{z+1} & a_z \end{array}}$$

$$\underset{\text{II}}{=} (-1)^{\mu} \boxed{\begin{array}{ccc} \mu & b_{\mu+1} & a_{n-\mu} \\ 2l_1 - \mu & s & z \\ n - \mu & a_z & b_{z+1} \end{array}}$$

$$\underset{\text{IV}}{=} (-1)^{z+\mu} \boxed{\begin{array}{ccc} b_{\mu+1} & \mu & a_{n-\mu} \\ s & 2l_1 - \mu & z \\ a_z & n - \mu & b_{z+1} \end{array}}$$

$$\underset{\text{II}}{=} (-1)^{z+\mu} \boxed{\begin{array}{ccc} b_{\mu+1} & a_{n-\mu} & \mu \\ a_z & b_{z+1} & n - \mu \\ s & z & 2l_1 - \mu \end{array}}$$

$$\underset{\text{I}}{=} (-1)^{z+\mu} \boxed{\begin{array}{ccc} b_{\mu+1} & a_z & s \\ a_{n-\mu} & b_{z+1} & z \\ \mu & n - \mu & 2l_1 - \mu \end{array}}. \tag{17}$$

5. REGGE SQUARE

Analogous to the example of the last section, one may verify that, with $s = n - z$,

$$\boxed{\begin{array}{ccc} \mu & a_{n-\mu} & b_{\mu+1} \\ 2l_1 - \mu & z & s \\ n - \mu & b_{z+1} & a_z \end{array}} = (-1)^{\mu+z} \boxed{\begin{array}{ccc} n - \mu & b_{z+1} & a_z \\ \mu & a_{n-\mu} & b_{\mu+1} \\ 2l_1 - \mu & z & s \end{array}}$$

$$= (-1)^{s} \boxed{\begin{array}{ccc} n - \mu & b_{z+1} & a_z \\ 2l_1 - \mu & z & s \\ \mu & a_{n-\mu} & b_{\mu+1} \end{array}}. \tag{18}$$

That is,[19]

$$C_{\text{Racah}} = C(\mu) = (-1)^{s} C(n - \mu). \tag{18'}$$

It is also a symmetry relation in the usual sense of the term. Now, but for a phase factor, the square on the right-hand side of (18) is the Regge square[10] expressed in our parameters. With this square, Regge denotes the $3j$ symbol

$$\begin{Bmatrix} l_1 & l_2 & j \\ m_1 & m_2 & -M \end{Bmatrix} = \frac{(-1)^{l_1 - l_2 + M}}{(2j+1)^{\frac{1}{2}}} C_{\text{Racah}}.$$

Apparently, there is just a difference of a phase factor between our square $C(n - \mu)$ and that of Regge. However, we feel that from the point of view of q.b. representations of CGc our symbolism (14), i.e.,

$$C_{\text{Racah}} = C(\mu) = C(s) = C(b_{z+1}),$$

is more consistent than that of Regge, since these are the only q.b. representations obtainable *directly* from the Racah formula[16] by using just definitions (2) and (3). This is immediately evident from the summation part

$$\sum_{y} [y!(\mu - y)!(s - y)!(b_{z+1} - y)!(z - \mu + y)! \times (a_z - \mu + y)!]^{-1}$$

[15] See. Ref. 6, formula (3.14).
[16] G. Racah, Phys. Rev. 62, 438 (1942), especially p. 440, formula (16).
[17] See Ref. 6, Eq. (5.7').
[18] Reference 6, Sec. 7, Table 1.
[19] Reference 6, formula (3.20).

of the Racah formula in our notation. On the other hand, $C(n-\mu)$ can only be gotten from $\sum_y$ by letting $y \to s - y$,[20] which gives rise to a phase factor $(-1)^s$, in accordance with (18'). In our way one is able to handle both the formulas of CGc and their symmetry relations directly, rather than through the intermediary 3-j symbol.

[20] Note, by (15), $s = n - z$.

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Quasibinomial Representations of Clebsch–Gordan Coefficients. II. "Negative" Representations*

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(Received 27 October 1967)

New quasibinomial forms are derived from the quasibinomial forms given previously by making use of both positive and negative generalized powers. They turn out to be a new representation of the Wigner-type unsymmetrical formulas of Clebsch–Gordan coefficients for angular momenta. Consequently, formulas of Racah, Majumdar, and Shimpuku are deduced as special cases. Rules to construct a square symbol are given from which all these "negative" quasibinomial representations or, more precisely, expansions can be read off directly. Thus, a unified treatment of both symmetrical and unsymmetrical formulas of Clebsch–Gordan coefficients is thereby accomplished.

I. INTRODUCTION

Utilizing the positive generalized powers,[1] it has been shown recently that the various symmetrical formulas of Clebsch–Gordan coefficients (CGc) for angular momenta can be treated in a unified manner.[2] In particular, the summation occurring in our basic CGc formula[3] was transformed into the so-called quasibinomial (q.b.) expansion,[2,4,5] a general one defined by

$$((ax - by))^{(n)} \equiv \sum_\alpha (-1)^\alpha \binom{n}{\alpha} a^{(n-\alpha)} x^{(n-\alpha)} b^{(\alpha)} y^{(\alpha)}, \quad (1)$$

where the generalized power[6] (g.p.)

$$x^{(n)} = x(x-1)(x-2)\cdots(x-n+1) = \binom{x}{n} n!, \quad x \geq n. \quad (2)$$

Note the following property:

$$((ax - by))^{(n)} = (-1)^n((by - ax))^{(n)}. \quad (2')$$

We recall that the main difficulty in carrying out the summation in Eq. (1) arose from the occurrence of $(-1)^\alpha$, which does not permit one to apply the Vandermonde binomial expansion of the generalized powers,[7] viz.,

$$(x + y)^{(n)} = \sum_\alpha \binom{n}{\alpha} x^{(n-\alpha)} y^{(\alpha)}. \quad (3)$$

In the following, we show that with the aid of our extension of (3) to the negative generalized powers, defined by

$$(-x)^{(n)} = (-1)^n (x + n - 1)^{(n)} \equiv (x^-)^{(n)}, \quad (4)$$

$(-1)^\alpha$ can be eliminated, thus enabling one to sum up at least a part of the q.b. expansion with the aid of the Vandermonde formula (3). However, the use of this extension, more precisely formula (A3) in Appendix A, gives rise to another summation which, finally, can be expressed again as another q.b. expansion comprising, this time, negative g.p. also. Consequently, we call the resulting expression simply a *negative* q.b. representation of CGc. In fact, such q.b. forms

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[1] See definition (2) above.

[2] S. M. R. Ansari, Fortschr. Physik **15**, 729 (1967).

[3] See Ref. 2, formula (3.1); or Ref. 4, formula (3).

[4] S. M. R. Ansari, Nuovo Cimento **38**, 1883 (1965).

[5] S. M. R. Ansari, J. Math. Phys. 9, 1295 (1968) (preceding paper).

[6] For a résumé of generalized powers (factorials), we refer to Sec. 2 of Ref. 2. See also C. Jordan, *Calculus of Finite Differences* (Chelsea Publishing Co., New York, 1950).

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correspond to the Wigner-type unsymmetrical[8–10] or fractional[11] CGc formulas.

As previously stated,[2,5] a negative q.b. representation may also be symbolized by means of a "negative" square, which enables one to read off all possible negative q.b. representations directly from it.

The importance of introducing the negative q.b. representation lies actually not only in simplifying the numerical calculations of CGc, especially by computers, in which case the Wigner-type formulas prove to be better than Racah's symmetrical ones,[12] but also in unifying the derivation of CGc formulas as such in contradistinction to the hitherto treatment as found in the literature.[8–11] The various formulas derived therein are shown to be special cases of our treatment.

Incidentally, we also show that the same symmetry relations, as in case of symmetrical CGc formulas, i.e., positive q.b. representations,[2] hold true now also[13] and their derivation is not "tedious" at all.[14]

II. NEGATIVE QUASIBINOMIAL EXPANSION

Let us rewrite the positive q.b. expansion (1), using (2), in the following form:

$$((ax - by))^{(n)} = \sum_{\alpha} (-1)^{\alpha} \binom{b}{\alpha} n^{(\alpha)} y^{(\alpha)} a^{(n-\alpha)} x^{(n-\alpha)}. \quad (5)$$

We now factorize the g.p. of x and y by the product formula (A1) as follows:

$$x^{(n-\alpha)} = x^{(n-b)}(x + b - n)^{(b-\alpha)} \quad (6)$$

and

$$y^{(\alpha)} = y^{(n-a)}(y + a - n)^{(a-n+\alpha)}. \quad (7)$$

The g.p. of $(y + a - n)$ may then be expanded by the formula (A4), viz.,

$$(y + a - n)^{(a-n+\alpha)} = \frac{(-1)^{n-\alpha}}{a^{(n-\alpha)}} \sum_{\beta} (-1)^{\beta} \binom{n-\alpha}{\beta} (y + a - n + \beta)^{(a)}. \quad (8)$$

Therefore (5) with the aid of (6)–(8) can be reexpressed as

$$((ax - by))^{(n)} = (-1)^n x^{(n-b)} y^{(n-a)} \sum_{\beta} \frac{1}{\beta!} (y + a - n + \beta)^{(a)} (-1)^{\beta} \times \sum_{\alpha} \binom{b}{\alpha} n^{(\alpha)} (n-\alpha)^{(\beta)} (x + b - n)^{(b-\alpha)}. \quad (9)$$

We notice that $(-1)^{\alpha}$ has been eliminated from the summation over α. We can now make use of the Vandermonde formula (3), bearing in mind the relation

$$n^{(\alpha)}(n - \alpha)^{(\beta)} = n^{(\alpha+\beta)} = n^{(\beta)}(n - \beta)^{(\alpha)}, \quad (9')$$

due to (A1). Thus (9) goes into

$$((ax - by))^{(n)} = (-1)^n x^{(n-b)} y^{(n-a)} \sum_{\beta} (-1)^{\beta} \binom{n}{\beta} \times (y' + \beta)^{(a)} (x' + n - \beta)^{(b)}, \quad (10)$$

where we have put

$$y' = y + a - n \quad \text{and} \quad x' = x + b - n. \quad (10')$$

To express (10) in a quasibinomial form, we employ the following factorizations due to (A2):

$$(y' + \beta)^{(a)} = (y' + \beta)^{(\beta)} y'^{(a-n)} y^{(n-\beta)}$$

and

$$(x' + n - \beta)^{(b)} = (x' + n - \beta)^{(n-\beta)} x'^{(b-n)} x^{(\beta)}.$$

This gives us immediately, with $x' + 1 \equiv x_1'$, $y' + 1 \equiv y_1'$, and introducing the negative g.p. due to (4), in analogy with (1), the *negative* q.b. expansion, viz.,

$$((ax - by))^{(n)} = \sum_{\beta} (-1)^{\beta} \binom{n}{\beta} y^{(n-\beta)} (x_1'^{-})^{(n-\beta)} x^{(\beta)} (y_1'^{-})^{(\beta)} \equiv ((yx_1'^{-} - xy_1'^{-}))^{(n)}, \quad (11)$$

since, again by (A1) and definition (2),

$$y^{(n-a)}(y + a - n)^{(a-n)} = 1 = x^{(n-b)}(x + b - n)^{(b-n)}.$$

Now instead of expanding $(y + a - n)^{(a-n+\alpha)}$, as in (8), had we expanded $(x + b - n)^{(b-\alpha)}$ according to (A4), we would have

$$(x + b - n)^{(b-\alpha)} = \frac{(-1)^{\alpha}}{b^{(\alpha)}} \sum_{\beta} (-1)^{\beta} \binom{\alpha}{\beta} (x + b - n + \beta)^{(b)}. \quad (12)$$

Note that (12) can also be obtained directly from (8) by the substitutions: $y \rightleftharpoons x$, $a \rightleftharpoons b$, i.e., $y' \rightleftharpoons x'$ and $n - \alpha \rightleftharpoons \alpha$. However, one should bear in mind that $(-1)^n$ appearing in (8) was cancelled finally, as we introduced the negative g.p. according to the definition (4). Consequently, this time it should appear in the final expression. Hence, without repeating the whole procedure again, we find that

$$((ax - by))^{(n)} = (-1)^n ((xy_1'^{-} - yx_1'^{-}))^{(n)}, \quad (13)$$

which is also evident from the definition (11) directly, that is,[15]

$$((yx_1'^{-} - xy_1'^{-}))^{(n)} = (-1)^n ((xy_1'^{-} - yx_1'^{-}))^{(n)}. \quad (14)$$

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[8] E. P. Wigner, *Group Theory and its Application to Atomic Spectra* (Academic Press Inc., New York, 1959), p. 191.

[9] S. D. Majumdar, Progr. Theoret. Phys. (Kyoto) **20**, 798 (1958).

[10] G. Racah, Phys. Rev. **62**, 438 (1942), especially p. 440, formula (15).

[11] T. Shimpuku, J. Math. Anal. Appl. **7**, 397 (1963).

[12] See Ref. 10, p. 440, formula (16).

[13] This seems to be quite obvious. Nevertheless, the symmetry relations are usually derived from the symmetrical CGc formulas, as done by Racah, Ref. 10.

[14] This is remarked by M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957), p. 40.

[15] Compare Eq. (2').

Similarly, the substitution $x \rightleftharpoons a$ yields

$$((ax - by))^{(n)} = ((xa - by))^{(n)} = ((ya_1'^{-} - ay_1''^{-}))^{(n)}, \tag{15}$$

where we have put

$$a_1' \equiv a' + 1 \equiv (a + b - n + 1) \tag{16a}$$

and

$$y_1'' \equiv y'' + 1 \equiv y + x - n + 1. \tag{16b}$$

On the other hand, $b \rightleftharpoons y$ gives

$$((ax - by))^{(n)} = ((by_1''^{-} - xa_1'^{-}))^{(n)}, \tag{17}$$

while with $a \rightleftharpoons x$ and $y \rightleftharpoons b$ one obtains

$$((ax - by))^{(n)} = ((by_1'^{-} - ax_1'^{-}))^{(n)}. \tag{18}$$

Thus we find that, corresponding to every positive q.b. expansion, there exist four different negative q.b. expansions with the *same* exponent n.

Now let us consider the formula (18). Because of (14) and (2′), it is clear that

$$((ax_1'^{-} - by_1'^{-}))^{(n)} = ((by - ax))^{(n)},$$

from which, with the help of the substitutions (10′), one gets

$$((ax_1^{-} - by_1^{-}))^{(n)} = ((b(y - a + n) - a(x - b + n)))^{(n)}. \tag{19}$$

This is the converse of the case we treated in the beginning of this section: a transformation of a negative q.b. expansion into a positive q.b. expansion. In fact, a detailed proof of (19) can be constructed.[16] The method is just the opposite of the above: one expands some g.p. first of all by means of the Vandermonde formula (3); after a little algebraic manipulation, one is able to sum up a part of the resulting expression with the aid of our extension (A3) and obtains (19). We sketch this proof very briefly in Appendix B.

The four negative q.b. expansions, with the same exponent n, (11), (15), (17), and (18), are not the only ones into which $((ax - by))^{(n)}$ can be transformed. One can also deduce from the latter, by simple algebraic manipulation, another set of negative q.b. expansions differing in exponents. We sketch this proof in Appendix C. The result is

$$((ax - by))^{(n)} = (-1)^{a-n}(n!/a'!)y^{(n-a)}x^{(n-b)}((y'x_1^{-} - x'y_1^{-}))^{(a')}, \tag{20}$$

where again y' and x' are given by (10′) and a' by (16a). Now, as previously, while $y \rightleftharpoons b$ in (20) gives ($y'' = x + y - n$)

$$((ax - by))^{(n)} = (-1)^{a-n}(n!/y'!)b^{(n-a)}x^{(n-y)} \times ((a'x_1^{-} - y''b_1^{-}))^{(y')}, \tag{21}$$

$a \rightleftharpoons x$, i.e., $a' \rightleftharpoons x'$, yields

$$((ax - by))^{(n)} = (-1)^{x-n}(n!/x'!)y^{(n-x)}a^{(n-b)} \times ((y''a_1^{-} - a'y_1^{-}))^{(x')}. \tag{22}$$

Finally, with $a \rightleftharpoons x$ and $b \rightleftharpoons y$, i.e., $y' \rightleftharpoons x'$ and $a' \rightleftharpoons y''$, from (20) one obtains

$$((ax - by))^{(n)} = (-1)^{x-n}(n!/y''!)b^{(n-x)}a^{(n-y)} \times ((x'a_1^{-} - y'b_1^{-}))^{(y'')}. \tag{23}$$

Summarizing, we may say that a positive q.b. expansion can be converted into essentially five negative q.b. expansions with different exponents and four negative q.b. expansions with the same exponents. These negative q.b. expansions give, in fact, all possible unsymmetrical formulas for CGc. Consequently, in the following section we show that the unsymmetrical expressions for CGc found in the literature[8–11] can be deduced as special cases from our above formulas.

III. FORMULAS OF WIGNER, RACAH, MAJUMDAR, AND SHIMPUKU

In this section we use our previous notation.[17] Our treatment of the preceding section has been quite general up to now. We take a particular positive q.b. representation of CGc as the starting one, namely, $C(b_{\mu+1})$[18]:

$$\begin{aligned} C_z &= (-1)^{z+\mu}C(b_{\mu+1}) \\ &\equiv (-1)^{z+\mu}(\mathcal{N}'/b_{\mu+1}!) \\ &\quad \times \{a_{n-\mu}!\,[\mu^{(\mu-z)}/b_{z+1}^{(\mu-z)}]s^{(\mu-z)}a_z^{(\mu-b_{z+1})}\}^{\frac{1}{2}} \\ &\quad \times ((b_{z+1}(2l_1 - \mu) - (n - \mu)z))^{(b_{\mu+1})}. \end{aligned} \tag{24}$$

With the aid of the formulas derived in the preceding section,[19] the q.b. expansion in (24) yields

$$((b_{z+1}(2l_1 - \mu) - (n - \mu)z))^{(b_{\mu+1})}$$

$$\underset{(11)}{=} ((za_{n-\mu+1}^{-} - (2l_1 - \mu)\mu_1^{-}))^{(b_{\mu+1})} \tag{25}$$

$$\underset{(15)}{=} ((zs_1^{-} - b_{z+1}a_{z+1}^{-}))^{(b_{\mu+1})} \tag{26}$$

$$\underset{(17)}{=} (((n - \mu)a_{z+1}^{-} - (2l_1 - \mu)s_1^{-}))^{(b_{\mu+1})} \tag{27}$$

$$\underset{(18)}{=} (((n - \mu)\mu_1^{-} - b_{z+1}a_{n-\mu+1}^{-}))^{(b_{\mu+1})}. \tag{28}$$

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[16] S. M. R. Ansari, D.Sc. dissertation, University of Tübingen (1966) (unpublished), Appendix C.

[17] See Ref. 5, Eqs. (15) and (15′).

[18] It can be read off directly from the corresponding positive square symbol of CGc; see Ref. 5, Eq. (17), and def. (1); or Ref. 2, Eq. (5.8).

[19] We write the number of the formula used under the sign of equality. Hereafter we shall often employ this notation to save space. We use the notation $n - z = s$, $a_x = l_1 - l_2 + M + x$, and $b_{x+1} = 2l_1 - a_x$. For other relations, see Ref. 5, Eq. (15).

These formulas are obviously to be multiplied by the same factors which appear in front of the double parentheses in (24). Then (25) and (28) give just the q.b. representations of two of the Majumdar formulas,[9] the hypergeometric functions of which can be expressed in our parameters[17] and with the notation (4) as

$$u_m = {}_2F_1(b_{\mu+1}^-, (2l_1 - \mu)^-; (2j)^-; 1 - x)$$

and

$${}_2F_1\left(b_{\mu+1}^-, (n-\mu)^-; (2j)^-; \frac{x-1}{x}\right).$$

Similarly, one may confirm that (26) and (27), multiplied again by the factors in (24), yield the unsymmetrical CGc formulas of Racah[10] and Wigner,[8] respectively, when one converts all the generalized powers into factorials by (2). We would like to mention here that (27) is not the only q.b. representation of the Wigner formula. First, by (14), the double parentheses (27) can be written as

$$(-1)^{b_{\mu+1}}(((2l_1-\mu)s_1^- - (n-\mu)a_{z+1}^-))^{(b_{\mu+1})}. \quad (29)$$

Using the relations[20]

$$(2l_1-\mu)^{(b_{\mu+1}-\alpha)} \underset{(A1)}{=} (2l_1-\mu)^{(b_{n+1})}\, a_{n-\mu}^{(n-\mu-\alpha)},$$

$$(s_1^-)^{(b_{\mu+1}-\alpha)} \underset{(A1')}{=} (s_1^-)^{(b_{n+1})}(b_z^-)^{(n-\mu-\alpha)}$$

$$\underset{(4)}{=} (-1)^{b_{n+1}} b_{z+1}^{(b_{n+1})}(b_z^-)^{(n-\mu-\alpha)},$$

and

$$\binom{b_{\mu+1}}{\alpha}(n-\mu)^{(\alpha)} \underset{(2)}{=} \binom{n-\mu}{\alpha} b_{\mu+1}^{(\alpha)},$$

where α is the summation index of the q.b. expansion (29), one can easily recast (29) into the form

$$b_{z+1}^{(b_{n+1})}(2l_1-\mu)^{(b_{n+1})}(-1)^{n-\mu} \times ((a_{n-\mu}b_z^- - b_{\mu+1}a_{z+1}^-))^{(n-\mu)}.$$

When one takes into account the factors appearing in (24) and uses (14) again, finally[21] one gets

$$\begin{aligned} C_z &= (-1)^{z+\mu}C^-(b_{\mu+1}) = (-1)^s C^-(n-\mu) \\ &\equiv (-1)^s(\mathcal{N}'/(n-\mu)!) \\ &\quad\times \{(2l_1-\mu)!\,[\mu^{(\mu-s)}/z^{(\mu-s)}]a_z^{(\mu-s)}b_{z+1}^{(\mu-z)}\}^{\frac{1}{2}} \\ &\quad\times ((b_{\mu+1}a_{z+1}^- - a_{n-\mu}b_z^-))^{(n-\mu)}, \end{aligned} \quad (30)$$

from which just by inspection one can immediately tell that another q.b. representation of the Wigner formula would be $C^-(a_{n-\mu})$. Incidently, we have also set up above a relation between two negative q.b. representations of CGc, viz.,

$$C^-(b_{\mu+1}) = (-1)^{n-\mu}C^-(n-\mu).$$

This is, in fact, a symmetry relation exactly as in the "positive" case.[22]

For the sake of completion we give, in the following, other negative q.b. expansions obtained by means of formulas (20)–(23):

$$((b_{z+1}(2l_1-\mu)-(n-\mu)z))^{(b_{\mu+1})}$$

$$\underset{(20)}{\longrightarrow} ((\mu(2l_1-\mu+1)^- - a_{n-\mu}z_1^-))^{(s)},$$

$$z_1 \equiv z+1, \quad (31)$$

$$\underset{(21)}{\longrightarrow} ((s(2l_1-\mu+1)^- - a_z(n-\mu+1)^-))^{(\mu)} \quad (32)$$

$$\underset{(22)}{\longrightarrow} ((a_z b_z^- - s z_1^-))^{(a_{n-\mu})} \quad (33)$$

$$\underset{(23)}{\longrightarrow} ((a_{n-\mu}b_z^- - \mu(n-\mu+1)^-))^{(a_z)}. \quad (34)$$

One may verify that the factors in front of these negative q.b. expansions in a complete formula $C^-(s)$, $C^-(\mu)$, $C^-(a_{n-\mu})$, and $C^-(a_z)$ turn out to be exactly the same as those appearing in the corresponding positive q.b. representations $C(s)$, $C(\mu)$, $C(a_{n-\mu})$, and $C(a_z)$, respectively.[23] This is a special feature of only q.b. representations of CGc expressions. Then, for instance, formula (32) is the q.b. representation of the Shimpuku fractional formula.[24] Incidentally, we have also shown the algebraic equivalence of various Wigner-type or unsymmetrical formulas for CGc, which, for instance, in terms of hypergeometric functions, is quite difficult.[25]

IV. NEGATIVE CGc SQUARE

Analogous to the positive CGc square,[2,5] one may symbolize the negative q.b. representations of CGc formulas by "negative" CGc squares, defined below, from which the same formulas of CGc can be read off *directly*. However, as mentioned above, the front factors of the positive and negative q.b. expansions with the *same* exponent being identical, one may confine oneself to reading only a negative double parentheses symbolizing a negative q.b. expansion from a negative square directly. Obviously the front factors are then to be deduced from the corresponding positive square, as previously.[2,5]

[20] For verification, use the relations given above in Ref. 19.

[21] Here we employ the notation C^- (exponent) for the negative q.b. representation of CGc. Formula (30) was reported for the first time in the short note, Ref. 4, formula (5). With the aid of the rules given in the following section, it can be read off from the square (18) of Ref. 5.

[22] See Ref. 2, Sec. 7, Table 1. This shows that the derivation of the symmetry relations from Wigner-type formulas is also not "tedious" (Ref. 14). Only in terms of ordinary factorials are they really unpractical, as was also pointed out by Racah, Ref. 10.

[23] They are given in Ref. 2, formulas (3.21), (3.14), (5.7), and (5.3), respectively. One can also deduce them from the corresponding positive CGc squares by means of definition (1) and rules I–IV in Ref. 5.

[24] See Ref. 11, p. 414, last formula.

[25] Compare the remark of S. D. Majumdar, Ref. 9, p. 802.

To this end, let the positive CGc square be given in terms of the general parameters used in Sec. 2. That is,[26]

$$\boxed{\begin{array}{c|cc} n & y'' & a' \\ \hline x' & a & y \\ y' & b & x \end{array}} \leftrightarrow ((ax - by))^{(n)}. \tag{35}$$

A negative CGc square is defined as the square obtained from (35) such that two of its elements are negative. Two negative squares are then obtained from (35) by the following rule:

Rule I: Substitute the second and third elements of the first row (or first column), increased by 1, by their negative values.

Thus

$$\boxed{\begin{array}{c|cc} n & y_1''^{-} & a_1'^{-} \\ \hline x' & a' & y \\ y' & b & x \end{array}} \quad \text{and} \quad \boxed{\begin{array}{c|cc} n & y'' & a' \\ \hline x_1'^{-} & a & y \\ y_1'^{-} & b & x \end{array}}. \tag{36}$$

Then one can associate with each of these squares, contrary to (35), two negative double parentheses as follows:

Rule II: Take any two row (column) elements of the inner (dotted) square and construct their products with the negative elements appearing in the corresponding columns (rows) of the main square.[27]

Rule III: The first product in a negative double parentheses is always the one which contains the second element of the first row (or first column) of the inner square.

We denote the first and second pair in the negative double parentheses by solid and dotted arrows, respectively. Four negative q.b. expansions symbolized by double parentheses with the same exponent n are then obtained with the aid of the above rules just by inspecting (36). That is, the left square in (36) gives formulas (15) and (17), while the right square yields formulas (18) and (11). Thus our purpose is achieved. As a simple exercise one may read off the formulas (25)–(28) from the "negative" of the left square in (37) below.

To get the other set of formulas (20)–(23), it is not necessary to set up other rules. One may start from the positive squares with the first elements as a', y', etc.,[28] obtain from them the corresponding negative squares, and apply the above rules to get again four negative double parentheses for each of the exponents a', y', x', and y''. It is clear that one of these negative double parentheses must be the one which appears in the set (20)–(23). We illustrate this procedure by the following example:

Again, in terms of our actual parameters of the theory of addition of two angular momenta, the positive square symbolizing the q.b. representation $C(b_{\mu+1})$ is given by[29]

$$(-1)^{z+\mu}\boxed{\begin{array}{ccc} b_{\mu+1} & a_z & s \\ a_{n-\mu} & b_{z+1} & z \\ \mu & n-\mu & 2l_1-\mu \end{array}} = \boxed{\begin{array}{ccc} s & a_z & b_{\mu+1} \\ 2l_1-\mu & n-\mu & \mu \\ z & b_{z+1} & a_{n-\mu} \end{array}}, \tag{37}$$

where we have used rules II–IV of Ref. 5. Then one of the corresponding negative squares will be

$$\boxed{\begin{array}{ccc} s & a_z & b_{\mu+1} \\ (2l_1-\mu+1)^- & n-\mu & \mu \\ z_1^- & b_{z+1} & a_{n-\mu} \end{array}} \rightarrow ((\mu(2l_1-\mu+1)^- - a_{n-\mu}z_1^-))^{(s)}. \tag{38}$$

For the complete formula $C^-(s)$, the front factors are to be obtained from the positive square $C(s)$ on the right-hand side of (37), as mentioned before.[30]

In conclusion, we reiterate that the negative q.b. representations of the unsymmetrical CGc formulas (Wigner type) can also be manipulated in the same way as their counterpart, the (positive) q.b. representations of the symmetrical CGc formulas, in the sense of writing out at once the double parentheses symbols with any exponent according to one's choice. Besides that, the symmetry relations between the various

[26] For its construction, we use the property: The sum of elements in a row (column) is always an integer; all such sums are equal. See. Ref. 2 or 5.

[27] The product so constructed may be thought of as a column product (or row product) contrary to the cross product in (35).

[28] Evidently these positive squares can be obtained from (35) by our previous rules given in Ref. 5.

[29] See Ref. 5, Eq. (17).

[30] For that, use definition (1) given in Ref. 5. Similarly, one may verify formula (30), i.e., $C^-(n-\mu)$ by reading it off from the square (18) given in Ref. 5.

unsymmetrical CGc formulas can also be derived easily, thereby showing their algebraic equivalence.

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It is a pleasure to thank Professor G. Elwert for his kind interest and encouragement. A relevant conversation with Dr. W. McClure is also gratefully acknowledged.

APPENDIX A

We recall[2,6] the product formula obeyed by a generalized power defined by (2), viz.,

$$x^{(n)} = x^{(m)}(x-m)^{(n-m)}. \tag{A1}$$

It is easy to confirm that for the negative g.p. one has analogously

$$(x^-)^{(n)} = (x^-)^{(m)}[(x+m)^-]^{(n-m)}. \tag{A1$'$}$$

One can also derive a 3-factors product formula from (A1), viz.,

$$(x+\beta)^{(n)} = (x+\beta)^{(\beta)}x^{(n-\alpha)}(x-n+\alpha)^{(\alpha-\beta)}, \tag{A2}$$

which will be used extensively in this paper.

Now, let $y \to -y \equiv y^-$ in the expansion (3). Then, by definition (4),

$$(x-y)^{(n)} = \sum_\alpha (-1)^\alpha \binom{n}{\alpha} x^{(n-\alpha)}(y+\alpha-1)^{(\alpha)}, \tag{A3}$$

which is the extension of (3).

Let us consider the g.p. $n^{(\alpha)}$. Again, by (4)

$$\begin{aligned} n^{(\alpha)} &= (-n^-)^{(\alpha)} = (-1)^\alpha(-n+\alpha-1)^{(\alpha)} \\ &= (-1)^\alpha[(x-n+\alpha)-(x+1)]^{(\alpha)}. \end{aligned}$$

Expanding the right-hand side by the formula (A3), one obtains, first of all,

$$n^{(\alpha)} = (-1)^\alpha \sum_\beta (-1)^\beta \binom{\alpha}{\beta}(x-n+\alpha)^{(\alpha-\beta)}(x+\beta)^{(\beta)},$$

which, being multiplied by $x^{(n-\alpha)}$ on both sides and by virtue of the 3-factors formula (A2) yields

$$x^{(n-\alpha)} = \frac{(-1)^\alpha}{n^{(\alpha)}} \sum_\beta (-1)^\beta \binom{\alpha}{\beta}(x+\beta)^{(n)}. \tag{A4}$$

Thus we get a series expansion of the generalized power $x^{(n-\alpha)}$.

APPENDIX B

We sketch in the following a few steps of the proof of formula (19). In fact, what we are doing is just recasting an unsymmetrical expression of CGc into a symmetrical one, in the usual sense of the term.[10] We think that effecting this transformation in terms of q.b. expansions and generalized powers is more elegant and clear than that in terms of factorials usually found in the literature.[31]

Analogous to a positive q.b. expansion (1), a negative q.b. expansion is, by definition ($x_1 \equiv x+1$, $y_1 \equiv y+1$):

$$\begin{aligned} ((ax_1^- - by_1^-))^{(n)} &= \sum_\alpha (-1)^\alpha \binom{n}{\alpha} a^{(n-\alpha)}(x_1^-)^{(n-\alpha)} \\ &\quad \times b^{(\alpha)}(y_1^-)^{(\alpha)} \\ &\underset{(4)}{=} (-1)^n \sum_\alpha (-1)^\alpha \binom{n}{\alpha}[a^{(n-\alpha)}(y+\alpha)^{(\alpha)}] \\ &\quad \times [(x+n-\alpha)^{(n-\alpha)}b^{(\alpha)}] \\ &\underset{(A2)}{=} (-1)^n \sum_\alpha (-1)^\alpha \binom{n}{\alpha} \frac{(y+\alpha)^{(y+n-a)}}{y^{(y-a)}} \\ &\quad \times \frac{(x+n-\alpha)^{(x+n-b)}}{x^{(x-b)}}. \end{aligned} \tag{B1}$$

Expanding now the g.p. of $(x+n-\alpha)$ with the aid of the Vandermonde formula (3) and using the relation (9′), (B1) may be expressed as

$$\begin{aligned} &((ax_1^- - by_1^-))^n \\ &= [y^{(y-a)}x^{(x-b)}]^{-1}(-1)^n \sum_\beta \binom{x+n-b}{\beta} x^{(x+n-b-\beta)}n^{(\beta)} \\ &\qquad \times \sum_\alpha (-1)^\alpha \binom{n-\beta}{\alpha}(y+\alpha)^{(y+n-a)}. \end{aligned} \tag{B2}$$

By (A2),

$$(y+\alpha)^{(y+n-a)} = (y+\alpha)^{(\alpha)}y^{(y-a+\beta)}(a-\beta)^{(n-\beta-\alpha)},$$

with which the summation over α in (B2) can be performed by (A3), to yield ($y_1 \equiv y+1$):

$$\sum_\alpha = (a-\beta-y_1)^{(n-\beta)} \underset{(4)}{=} (-1)^{n-\beta}(y+n-a)^{(n-\beta)}.$$

With the factorization

$$x^{(x+n-b-\beta)} = x^{(x-b)}b^{(n-\beta)}$$

and

$$y^{(y-a+\beta)} = y^{(y-a)}a^{(\beta)}$$

by (A1) and using the "exchange"

$$n^{(\beta)}\binom{x+n-b}{\beta} = \binom{n}{\beta}(x+n-b)^{(\beta)},$$

(B2) goes into (19) by definition (1). Q.E.D.

APPENDIX C

To prove formula (20), we proceed as follows. We rewrite the q.b. expansion (1) as

$$\begin{aligned} ((ax-by))^{(n)} &= \sum_\alpha (-1)^\alpha \binom{n}{n-\alpha} b^{(\alpha)}y^{(\alpha)}a^{(n-\alpha)}x^{(n-\alpha)} \\ &\underset{(2)}{=} \sum_\alpha (-1)^\alpha n^{(n-\alpha)}b^{(\alpha)}y^{(\alpha)}\binom{a}{n-\alpha}x^{(n-\alpha)}, \end{aligned} \tag{C1}$$

[31] See, for example, G. Racah, Phys. Rev. **61**, 187 (1942), especially the Appendix, p. 196.

and express the g.p. of degree α by (A1) as

$$b^{(\alpha)}y^{(\alpha)} = \alpha^{(2\alpha-b)}b^{(b-\alpha)}y^{(b-b+\alpha)}.$$

The generalized power of y in the above relation may now be expanded with the aid of (A4). With these, (C1) is converted into

$$((ax - by))^{(n)} = \sum_\beta \frac{(-1)^{\beta+b}}{\beta!}(y+\beta)^{(b)} \times \sum_\alpha \binom{a}{n-\alpha} x^{(n-\alpha)} \times [n^{(n-\alpha)}\alpha^{(2\alpha-b)}(b-\alpha)^{(\beta)}]. \quad \text{(C2)}$$

But by (A2), the square bracket in (C2) is equal to

$$n^{(n+\alpha-b+\beta)} \underset{\text{(A1)}}{=} n^{(n-a'+\beta)}(a'-\beta)^{(a-n+\alpha)},$$

where a' is given by (16a). Therefore one is able to perform the summation over α in (C2) by the Vandermonde formula (3), namely,

$$\sum_\alpha \binom{a}{n-\alpha} x^{(n-\alpha)}(a'-\beta)^{(a-n+\alpha)} = (x+a'-\beta)^{(a)}.$$

Equation (C2) may then be written as

$$((ax - by))^{(n)} = (-1)^b \frac{n!}{a'!}\sum_\beta (-1)^\beta \binom{a'}{\beta}(y+\beta)^{(b)}(x+a'-\beta)^{(a)}. \quad \text{(C3)}$$

Factorizing now the g.p. in (C3) by 3-factors formula (A2) as follows,

$$(y+\beta)^{(b)} = (y+\beta)^{(\beta)}y^{(n-a)}y'^{(a'-\beta)},$$
$$y' = y + a - n,$$
$$(x+a'-\beta)^{(a)} = (x+a'-\beta)^{(a'-\beta)}x^{(n-b)}x'^{(\beta)},$$
$$x' = x + b - n,$$

and using definition (4), (C3) is finally transformed into the negative q.b. form, viz.,

$$((ax - by))^{(n)} = (-1)^{a-n}\frac{n!}{a'!}y^{(n-a)}x^{(n-b)}((y'x_{\bar{1}} - x'y_{\bar{1}}))^{(a')}.$$

Q.E.D.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 9, NUMBER 8 AUGUST 1968

Construction of Invariants for Lie Algebras of Inhomogeneous Pseudo-Orthogonal and Pseudo-Unitary Groups*

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A method of constructing invariants for the Lie algebras of the inhomogeneous pseudo-orthogonal and pseudo-unitary groups from invariants of the (homogeneous) pseudo-orthogonal and pseudo-unitary groups, respectively, is presented. The method is based on the "expansion" (or "deformation") of the inhomogeneous algebras to homogeneous ones. Several examples are worked out.

1. INTRODUCTION

Herein is presented a method of constructing invariants for the Lie algebras of the inhomogeneous pseudo-orthogonal and pseudo-unitary groups from invariants of the Lie algebras of the (homogeneous) pseudo-orthogonal and pseudo-unitary groups, respectively.[1] The method is based on "expansion" (or "deformation") of the Lie algebra of $IO(p,q)$ or $IU(p,q)$ to the Lie algebras of $O(p+1,q)$ and $O(p,q+1)$, or $U(p+1,q)$ and $U(p,q+1)$, respectively, whereby the two homogeneous algebras in each case are constructed as subalgebras of the enveloping algebra of the inhomogeneous algebra.[2,3] Any invariant of the Lie algebra of $O(p+1,q)$, $O(p,q+1)$, $U(p+1,q)$, or $U(p,q+1)$ thus corresponds to an element of the enveloping algebra of $IO(p,q)$ or $IU(p,q)$, whichever is appropriate, and is shown to be an invariant of the Lie algebra of $IO(p,q)$ or $IU(p,q)$, respectively.

2. DEFINITIONS AND NOTATION

Greek indices run from 1 to n; Latin indices run from 1 to $n+1$. $g_{\mu\nu}$ is a diagonal metric tensor whose

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† On leave of absence from Tel-Aviv University.
[1] This general method is based on relations found in Ref. 2 [Eqs. (41), (129), and (132)], which are special cases of the method.
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But by (A2), the square bracket in (C2) is equal to

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Equation (C2) may then be written as

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eigenvalues consist of $+1$ p times and -1 q times, where $p+q=n$. g_{ab} is also diagonal with $g_{n+1n+1} = \pm 1$. The summation convention is assumed. The metric tensors raise and lower indices as usual.

$O(p,q)$: the n-dimensional (homogeneous) pseudo-orthogonal groups. $M_{\mu\nu} = -M_{\nu\mu}$ form a basis for their Lie algebras, where

$$[M_{\mu\nu}, M_{\rho\sigma}] = g_{\mu\rho}M_{\nu\sigma} - g_{\nu\rho}M_{\mu\sigma} + g_{\mu\sigma}M_{\rho\nu} - g_{\nu\sigma}M_{\rho\mu}. \quad (1)$$

$IO(p,q)$: the n-dimensional inhomogeneous pseudo-orthogonal groups. $M_{\mu\nu} = -M_{\nu\mu}$ and P_μ form a basis, where Eq. (1) holds and

$$[M_{\mu\nu}, P_\rho] = g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu, \quad (2)$$

$$[P_\mu, P_\nu] = 0. \quad (3)$$

$U(p,q)$: the n-dimensional (homogeneous) pseudo-unitary groups. A basis for their Lie algebras is formed by $E_{\mu\nu} = -E_{\nu\mu}$ and $F_{\mu\nu} = F_{\nu\mu}$ obeying

$$[E_{\mu\nu}, E_{\rho\sigma}] = g_{\mu\rho}E_{\nu\sigma} - g_{\nu\rho}E_{\mu\sigma} + g_{\mu\sigma}E_{\rho\nu} - g_{\nu\sigma}E_{\rho\mu}, \quad (4)$$

$$[E_{\mu\nu}, F_{\rho\sigma}] = g_{\mu\rho}F_{\nu\sigma} - g_{\nu\rho}F_{\mu\sigma} + g_{\mu\sigma}F_{\rho\nu} - g_{\nu\sigma}F_{\rho\mu}, \quad (5)$$

$$[F_{\mu\nu}, F_{\rho\sigma}] = g_{\mu\rho}E_{\nu\sigma} + g_{\nu\rho}E_{\mu\sigma} - g_{\mu\sigma}E_{\rho\nu} - g_{\nu\sigma}E_{\rho\mu}. \quad (6)$$

$IU(p,q)$: the n-dimensional inhomogeneous pseudo-unitary groups. $E_{\mu\nu} = -E_{\nu\mu}$, $F_{\mu\nu} = F_{\nu\mu}$, Q_μ, and R_μ form a basis for their Lie algebras, where Eqs. (4)–(6) hold and

$$[E_{\mu\nu}, Q_\rho] = g_{\mu\rho}Q_\nu - g_{\nu\rho}Q_\mu, \quad (7)$$

$$[E_{\mu\nu}, R_\rho] = g_{\mu\rho}R_\nu - g_{\nu\rho}R_\mu, \quad (8)$$

$$[F_{\mu\nu}, Q_\rho] = -g_{\mu\rho}R_\nu - g_{\nu\rho}R_\mu, \quad (9)$$

$$[F_{\mu\nu}, R_\rho] = g_{\mu\rho}Q_\nu + g_{\nu\rho}Q_\mu, \quad (10)$$

$$[Q_\mu, Q_\nu] = [Q_\mu, R_\nu] = [R_\mu, R_\nu] = 0. \quad (11)$$

3. PSEUDO-ORTHOGONAL GROUPS

$P^2 = P_\mu P^\mu$ is a second-degree invariant of $IO(p,q)$. Denote

$$J_\mu = \tfrac{1}{2}(P^\nu M_{\mu\nu} + M_{\mu\nu}P^\nu), \quad (12)$$

and define $M_{ab} = -M_{ba}$ by

$$M_{\mu n+1} = (-\epsilon P^2)^{-\frac{1}{2}} J_\mu, \quad (13)$$

with

$$g_{n+1n+1} = \epsilon, \quad (14)$$

where $\epsilon = \pm 1$. The M_{ab} obey

$$[M_{ab}, M_{cd}] = g_{ac}M_{bd} - g_{bc}M_{ad} + g_{ad}M_{cb} - g_{bd}M_{ca}, \quad (15)$$

and therefore form a basis of the Lie algebra of $O(p+1,q)$ ($\epsilon = +1$) or $O(p,q+1)$ ($\epsilon = -1$).[2–4]

[4] This construction was performed for $IO(p,q)$ by A. Sankaranarayanan, Nuovo Cimento **38**, 1441 (1965), and for $IO(3,1)$ by M. A. Melvin, Bull. Am. Phys. Soc. 7, 493 (1962), and by M. Y. Han, Nuovo Cimento **42B**, 367 (1966). See also A. Böhm, Phys. Rev. **145**, 1212 (1966) and C. Fronsdal, Rev. Mod. Phys. **37**, 211 (1965).

We now prove the following:

Theorem: If an element of the enveloping algebra of $IO(p,q)$ (i.e., a polynomial in $M_{\mu\nu}$ and P_μ) commutes with all $M_{\mu\nu}$ and J_μ, it also commutes with all P_μ.

Proof: Let X be a jth-degree polynomial in $M_{\mu\nu}$ and P_μ obeying

$$[M_{\mu\nu}, X] = [J_\mu, X] = 0 \quad (16)$$

for all μ, ν. Denote

$$Y_\mu = [P_\mu, X]. \quad (17)$$

Each component of Y_μ either is a jth-degree polynomial or vanishes. Equations (12), (16), and (17) give

$$M_{\mu\nu}Y^\nu + Y^\nu M_{\mu\nu} = 0. \quad (18)$$

From Eqs. (2), (17), and the Jacobi identity for commutators, we obtain

$$[M_{\mu\nu}, Y^\nu] = (1-n)Y_\mu. \quad (19)$$

Addition of Eqs. (18) and (19) gives

$$2M_{\mu\nu}Y^\nu = (1-n)Y_\mu. \quad (20)$$

Now if any of the Y_μ are assumed to be nonzero, Eq. (20) can be used to show contradictions such as the assumed nonzero component vanishing or a jth-degree polynomial equaling a $(j+1)$th degree polynomial. So we have the result that $Y_\mu = 0$ for all μ. Q.E.D.

Take any invariant of the Lie algebra of $O(p+1,q)$ or $O(p,q+1)$ of the form $M^{a_1}{}_{a_2}M^{a_2}{}_{a_3}\cdots M^{a_m}{}_{a_1}$ (m even and $<n+1$) or $\epsilon_{a_1\cdots a_{n+1}}M^{a_1a_2}\cdots M^{a_na_{n+1}}$ for $n+1$ even ($\epsilon_{a_1\cdots a_{n+1}}$ completely antisymmetric with $\epsilon_{1\cdots n+1} = 1$). Substitute Eqs. (13), (14), and (12). Either all square roots and ϵ's disappear or they can be factored out. Multiply by ϵ and $(-\epsilon P^2)^{\frac{1}{2}}$, if necessary, and by P^2 to the lowest power sufficient to remove all negative powers of P^2. (The expression can also be multiplied by any convenient numerical factor.) The resulting polynomial in $M_{\mu\nu}$ and P_μ commutes with all $M_{\mu\nu}$ and J_μ and, according to the above theorem, therefore commutes with all P_μ and is an invariant of the Lie algebra of $IO(p,q)$. In specific cases this resulting $IO(p,q)$ invariant might reduce to a function of lower-degree invariants.

Example 1^2: Starting with the second-degree invariant $M^a{}_bM^b{}_a$, we obtain the fourth (in general)-degree invariant

$$P^2M^\mu{}_\nu M^\nu{}_\mu - 2P^\mu P^\nu M_{\mu\sigma}M^\sigma{}_\nu - \tfrac{1}{2}(n-1)^2P^2.$$

For $n=3$ this becomes $(\epsilon_{\mu\sigma\rho}P^\mu M^{\nu\rho})^2 + 4P^2$ and is a function of second-degree invariants.

For $n = 4$ we obtain $\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}\epsilon_{\mu\alpha\beta\gamma}P^\alpha M^{\beta\gamma} + 9P^2$, which is a *bona fide* fourth-degree invariant.

Example 2: For $n = 3$ the second-degree invariant $\epsilon_{abcd}M^{ab}M^{cd}$ gives the second-degree invariant $\varepsilon_{\mu\nu\rho} P^\mu M^{\nu\rho}$.

Example 3: The fourth-degree invariant $M^a{}_b M^b{}_c M^c{}_d M^d{}_a$ gives rise to the eighth-(in general)-degree invariants

$$\begin{aligned}P^4 M^\alpha{}_\beta M^\beta{}_\gamma M^\gamma{}_\delta M^\delta{}_\alpha &- 4P^2P^\nu P^\mu M_{\mu\alpha}M^\alpha{}_\beta M^\beta{}_\gamma M^\gamma{}_\nu \\ &+ 2P^\sigma P^\rho P^\nu P^\mu M_{\mu\alpha}M^\alpha{}_\nu M_{\rho\beta}M^\beta{}_\sigma \\ &- (3n-4)P^4 M^\alpha{}_\beta M^\beta{}_\alpha \\ &+ (n-1)(3n-2)P^2P^\nu P^\mu M_{\mu\alpha}M^\alpha{}_\nu \\ &- \tfrac{1}{8}(n-1)^3(n-3)P^4.\end{aligned}$$

4. PSEUDO-UNITARY GROUPS

$Q^2 + R^2 = Q_\mu Q^\mu + R_\mu R^\mu$ is a second-degree invariant of $IU(p, q)$. Denote

$$A_\mu = \tfrac{1}{2}(Q^\nu E_{\mu\nu} + E_{\mu\nu}Q^\nu - R^\nu F_{\mu\nu} - F_{\mu\nu}R^\nu), \quad (21)$$

$$B_\mu = \tfrac{1}{2}(R^\nu E_{\mu\nu} + E_{\mu\nu}R^\nu + Q^\nu F_{\mu\nu} + F_{\mu\nu}Q^\nu), \quad (22)$$

$$\begin{aligned}\Lambda &= \tfrac{1}{2}(Q^\mu B_\mu + B_\mu Q^\mu - R^\mu A_\mu - A_\mu R^\mu) \\ &= 2Q^\mu R^\nu E_{\mu\nu} + (Q^\mu Q^\nu + R^\mu R^\nu)F_{\mu\nu},\end{aligned} \quad (23)$$

and define $E_{ab} = -E_{ba}$ and $F_{ab} = F_{ba}$ by

$$E_{\mu n+1} = [-\epsilon(Q^2 + R^2)]^{-\frac{1}{2}}A_\mu, \quad (24)$$

$$F_{\mu n+1} = [-\epsilon(Q^2 + R^2)]^{-\frac{1}{2}}B_\mu, \quad (25)$$

$$F_{n+1 n+1} = [-\epsilon(Q^2 + R^2)]^{-1}\Lambda, \quad (26)$$

with

$$g_{n+1n+1} = \epsilon, \quad (27)$$

where $\epsilon = \pm 1$. The E_{ab} and F_{ab} obey Eqs. (4)–(6) with μ, ν, ρ, and σ replaced by a, b, c, and d, respectively, and therefore form a basis of the Lie algebra of $U(p+1, q)$ ($\epsilon = +1$) or $U(p, q+1)$($\varepsilon = -1$).[2,3]

We now prove the following:

Theorem: If an element of the enveloping algebra of $IU(p, q)$ (i.e., a polynomial in $E_{\mu\nu}$, $F_{\mu\nu}$, Q_μ, and R_μ) commutes with all $E_{\mu\nu}$, $F_{\mu\nu}$, A_μ, and B_μ, it also commutes with all Q_μ and R_μ.

Proof: Following the line of the proof of the previous section, let X be a jth-degree polynomial in $E_{\mu\nu}$, $F_{\mu\nu}$, Q_μ, and R_μ obeying

$$[E_{\mu\nu}, X] = [F_{\mu\nu}, X] = [A_\mu, X] = [B_\mu, X] = 0 \quad (28)$$

for all μ, ν. Denote

$$Y_\mu = [Q_\mu, X], \quad (29)$$

$$Z_\mu = [R_\mu, X]. \quad (30)$$

Each component of Y_μ and Z_μ either is a jth-degree polynomial or vanishes. Equations (21), (22), and (28)–(30) give

$$E_{\mu\nu}Y^\nu + Y^\nu E_{\mu\nu} - F_{\mu\nu}Z^\nu - Z^\nu F_{\mu\nu} = 0, \quad (31)$$

$$E_{\mu\nu}Z^\nu + Z^\nu E_{\mu\nu} + F_{\mu\nu}Y^\nu + Y^\nu F_{\mu\nu} = 0. \quad (32)$$

From Eqs. (7)–(10), (29), (30), and the Jacobi identity for commutators, we obtain

$$[E_{\mu\nu}, Y^\nu] = (1-n)Y_\mu, \quad (33)$$

$$[E_{\mu\nu}, Z^\nu] = (1-n)Z_\mu, \quad (34)$$

$$[F_{\mu\nu}, Y^\nu] = -(1+n)Z_\mu, \quad (35)$$

$$[F_{\mu\nu}, Z^\nu] = (1+n)Y_\mu. \quad (36)$$

Subtraction of Eq. (36) from the sum of Eqs. (31) and (33) gives

$$E_{\mu\nu}Y^\nu - F_{\mu\nu}Z^\nu = -nY_\mu, \quad (37)$$

and addition of Eqs. (32), (34), and (35) gives

$$E_{\mu\nu}Z^\nu + F_{\mu\nu}Y^\nu = -nZ_\mu. \quad (38)$$

Now if any of the Y_μ and Z_μ are assumed to be nonzero, Eqs. (37) and (38) can be used to show contradictions such as the assumed nonzero component vanishing or a jth degree polynomial equaling a $(j+1)$th-degree polynomial. Therefore $Y_\mu = Z_\mu = 0$ for all μ. Q.E.D.

Take any invariant of the Lie algebra of $U(p+1, q)$ or $U(p, q+1)$ having the form of a polynomial whose terms are proportional to products of G_{ab}'s and H_{ab}'s with all indices contracted among themselves. Substitute Eqs. (24)–(27) and (21)–(23). All square roots and ϵ's disappear. Multiply by $Q^2 + R^2$ to the lowest power sufficient to remove all negative powers of $Q^2 + R^2$ (and by any convenient numerical factor, if desired). The resulting polynomial in $E_{\mu\nu}$, $F_{\mu\nu}$, Q_μ, and R_μ commutes with all $E_{\mu\nu}$, $F_{\mu\nu}$, A_μ, and B_μ; according to the above theorem, it therefore commutes with all Q_μ and R_μ and is an invariant of the Lie algebra of $IU(p, q)$. In specific cases this resulting $IU(p, q)$ invariant might reduce to a function of lower-degree invariants.

Example 1²: Starting with the first-degree invariant $F_a{}^a$, we obtain the third (in general)-degree invariant $(Q^2 + R^2)F_\alpha{}^\alpha - \Lambda$.

Example 2²: The second-degree invariant $E^a{}_b E^b{}_a - F^a{}_b F^b{}_a$ gives the sixth (in general)-degree invariant

$$\begin{aligned}(Q^2 + R^2)^2(E^\alpha{}_\beta E^\beta{}_\alpha - F^\alpha{}_\beta F^\beta{}_\alpha) \\ + 2(Q^2 + R^2)(A_\alpha A^\alpha + B_\alpha B^\alpha) - \Lambda^2.\end{aligned}$$

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Current Distribution in a Thin Superconducting Strip*

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A mathematical model of the alternating-current distribution for a long superconducting strip with small thickness is analyzed. The interesting feature of the model is that it is soluble. The current distribution is found from the analytic solution to a singular integral equation, and from this solution it is possible to predict the behavior of the distribution throughout a complete cycle. Previous (unpublished) work showed that purely physical arguments did not give the correct predictions for this behavior at the end of each half-cycle. The present model overcomes this deficiency.

1. INTRODUCTION

When alternating current flows in a thin superconducting strip, the alternating magnetic field associated with the current cuts into the plane of the strip and the current tries to distribute itself across the width so as to make this field zero. Perfect flux exclusion right to the edges would entail infinite current per unit width at the edges (and finite values elsewhere in the width). In fact, the current per unit width i is limited by the properties of the superconductor to a critical value i_0, which is here simplified to a constant independent of the field.

Then there is always a portion of the strip from each edge inward which carries i_0, and this portion increases with current. It can be shown that a hysteretic effect takes place and that, when the current is reduced after its initial maximum, the regions do not shrink but instead a region of opposite polarity grows inward from each edge. In the inner part of the strip the current adjusts itself to make the field there zero, but in the outer regions carrying i_0 this is not possible; flux enters the strip and causes a power loss because the voltages which it induces are in the same direction in each part of the region as is the current.

In Fig. 1 the width of the strip is represented by $2a$ and the x axis is chosen so that $-a \leq x \leq a$. The current distribution $i(x)$ at some stage in the half-cycle can be approximated to the form shown. The inner portion of the strip $-b \leq x \leq b$ is superconducting, i.e., its resistivity remains zero, and no flux enters. The jumps in $i(x)$ at D, A of Fig. 1 (where $AB = CD = \alpha$) are here represented vertically in the following analysis, but a truer representation near these points would require the introduction of a rapidly increasing function. In symbols the above conditions on $i(x)$ now become

$$i(x) = -i_0, \begin{cases} -a < x < -a + \alpha, & (1) \\ a - \alpha < x < a, & (2) \end{cases}$$

$$i(x) = i_0, \begin{cases} -a + \alpha < x < -b, & (3) \\ b < x < a - \alpha. & (4) \end{cases}$$

We impose the condition

$$i(x) = i_0 \quad \text{at} \quad x = \pm b. \tag{5}$$

The starting point for the development of the theory is the London equation for the current density $\mathbf{i}$, $\operatorname{curl} \mathbf{i} = -\beta^2 \mathbf{H}$, and the Maxwell equation $\operatorname{curl} \mathbf{H} = \mathbf{i}$. Here $1/\beta$ is the penetration depth. Since the problem is one-dimensional, it is straightforward to see that we can represent the magnetic-field distribution $H(\zeta)$ at some point ζ by means of

$$H(\zeta) = (2\pi\beta^4)^{-1}\, P \int_{-a}^{a} \frac{i(x)}{\zeta - x}\, dx, \tag{6}$$

where the P signifies that the Cauchy principal value of the integral has to be taken. The representation (6) satisfies the one-dimensional forms of the London and Maxwell equations. The reader may note that Marcus[1] considered a problem similar to the above, although he dealt with direct-current flow and considered a cylindrical superconductor. The above representation (6) can be deduced from the formulas given by Marcus. The singular integral equation for the current density derived by Marcus is of a mathematically more complex form than Eq. (14) below.

The magnetic field has to be zero over the superconducting portion of the strip and hence

$$P \int_{-a}^{a} \frac{i(x)}{\zeta - x}\, dx = 0, \quad -b \leq \zeta \leq b. \tag{7}$$

There is now sufficient information available to enable

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[1] P. M. Marcus, Intern. Conf. Low Temp. Phys., Toronto 7, 418 (1960).

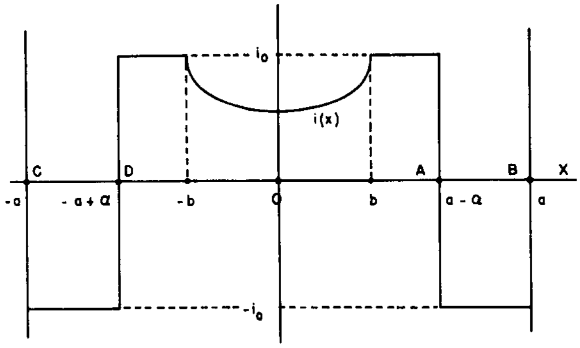

FIG. 1. Current distribution across the strip.

a representation to be discovered for the current distribution over the superconducting portion of the strip. To facilitate interpretation of the integrals which arise when dealing with (7), we introduce the quantity $i'(x)$ defined as

$$i'(x) = i_0 - i(x) \quad \text{for} \quad -b \le x \le b. \tag{8}$$

Thus, in place of (5), we now have

$$i'(x) = 0 \quad \text{at} \quad x = \pm b. \tag{9}$$

The remaining sections concern the analysis of the consequential current distribution.

2. SOLUTION OF THE INTEGRAL EQUATION

Application of (7) and (1)–(4), for $-b \le \zeta \le b$, now yields

$$P\int_{-b}^{b} \frac{i'(x)}{\zeta - x}\, dx = i_0 F(\zeta), \quad -b \le \zeta \le b, \tag{10}$$

where we have set

$$F(\zeta) = 2\ln|\zeta + a - \alpha| - 2\ln|\zeta - a + \alpha| - \ln|\zeta + a| + \ln|\zeta - a|. \tag{11}$$

To reduce the algebra we introduce the transformations

$$(x + b)/2b = u, \quad (\zeta + b)/2b = v, \tag{12}$$

and write

$$I(u) = i'(2bu - b)/i_0. \tag{13}$$

This means that in place of (10) we now have

$$P\int_0^1 \frac{I(u)}{u - v}\, du = -f(v), \quad 0 \le v \le 1, \tag{14}$$

where now we have defined

$$f(v) = F(2bv - b). \tag{15}$$

Equation (10) is a singular integral equation with a Cauchy kernel for the quantity $I(u)$. Various authors have examined the solution of equations of this type and, in particular, S. G. Mikhlin[2] has devoted some effort in this direction; from the analysis in his book we can place the solution of (10) in the concise form

$$I(u) = \frac{-1}{\pi^2}\left(\frac{u}{1-u}\right)^{\frac{1}{2}} \times P\int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{f(v)}{u - v}\, dv + \frac{c}{[u(1-u)]^{\frac{1}{2}}}, \tag{16}$$

where c is an arbitrary constant. By simple rearrangement,

$$I(u) = \frac{-1}{\pi^2}\left(\frac{u}{1-u}\right)^{\frac{1}{2}} \int_0^1 \frac{f(v)\, dv}{[v(1-v)]^{\frac{1}{2}}} - \frac{1}{\pi^2}[u(1-u)]^{\frac{1}{2}} P\int_0^1 \frac{f(v)\, dv}{(u - v)[v(1-v)]^{\frac{1}{2}}} + \frac{c}{[u(1-u)]^{\frac{1}{2}}}. \tag{17}$$

It is possible to show that the first integral in (16) has the value zero and hence we have from (17)

$$I(u) = \frac{-1}{\pi^2}[u(1-u)]^{\frac{1}{2}} P\int_0^1 \frac{f(v)\, dv}{(u - v)[v(1-v)]^{\frac{1}{2}}} + \frac{c}{[u(1-u)]^{\frac{1}{2}}}. \tag{18}$$

When u is near zero (or unity), it can be shown that the first integral in (18) is bounded; consequently,

$$\lim_{u \to 0 \text{ or } 1} [u(1-u)]^{\frac{1}{2}} P\int_0^1 \frac{f(v)\, dv}{(u - v)[v(1-v)]^{\frac{1}{2}}} = 0.$$

Also application of condition (9), via the transformation (13), yields

$$I(0) = I(1) = 0.$$

It follows from the above analysis that we can take the constant c to be zero in (18); thus the solution (which is now unique) for $I(u)$ is

$$I(u) = \frac{-1}{\pi^2}[u(1-u)]^{\frac{1}{2}} P\int_0 \frac{f(v)\, dv}{(u - v)[v(1-v)]^{\frac{1}{2}}}, \quad 0 \le u \le 1. \tag{19}$$

3. ANALYSIS OF THE SOLUTION

On using (12), we now put (15) in the form

$$f(v) = 2\ln\left|\frac{2bv - b + a - \alpha}{2bv - b - a + \alpha}\right| + \ln\left|\frac{2bv - b - a}{2bv - b + a}\right|.$$

If we set $v = \sin^2\phi$, this expression becomes

$$f(\sin^2\phi) = 2\ln\left|\frac{a - \alpha - b\cos 2\phi}{a - \alpha + b\cos 2\phi}\right| - \ln\left|\frac{a - b\cos 2\phi}{a + b\cos 2\phi}\right|.$$

[2] S. G. Mikhlin, *Integral Equations* (Pergamon Press, Inc., New York, 1957).

From (19), if we introduce $v = \sin^2\phi$, $u = \sin^2\psi$, then we have

$$I(\sin^2\psi) = \frac{-4}{\pi^2}\sin\psi\cos\psi \times P\int_0^{\pi/2}\frac{\ln|(a-\alpha-b\cos 2\phi)/(a-\alpha+b\cos 2\phi)|\,d\phi}{\sin^2\psi-\sin^2\phi} + \frac{2}{\pi^2}\sin\psi\cos\psi \times P\int_0^{\pi/2}\frac{\ln|(a-b\cos 2\phi)/(a+b\cos 2\phi)|\,d\phi}{\sin^2\psi-\sin^2\phi}.$$

Let $K(\psi)$ denote the integral

$$P\int_0^{\pi/2}\frac{\ln|(A-B\cos 2\phi)/(A+B\cos 2\phi)|\,d\phi}{\sin^2\psi-\sin^2\phi}.$$

Then we have

$$K(\psi) = P\int_0^{\pi/2}\frac{\sec^2\phi\sec^2\psi}{\tan^2\psi-\tan^2\phi} \times \ln\left[\frac{A-B+(A+B)\tan^2\phi}{A+B+(A-B)\tan^2\phi}\right]d\phi,$$

if $A-B$ and $A+B$ are each greater than zero. In this integral, use the substitutions $x = \tan\phi$, $r = \tan\psi$:

$$R(r) = (1+r^2)\,P\int_0^{\infty}\frac{1}{r^2-x^2} \times \ln\left[\frac{A-B+(A+B)x^2}{A+B+(A-B)x^2}\right]dx,$$

where we have set $R(r) = K(\tan^{-1} r)$. Now, by simple analysis, we have

$$\ln\{A-B+(A+B)x^2\} = \ln(A+B) + \ln\left\{1+\left(\frac{A-B}{A+B}\right)\frac{1}{x^2}\right\} + \ln x^2.$$

If we place $k^2 = (A-B)/(A+B)$, we may now write $R(r)$ in the form

$$R(r) = (1+r^2)P\int_0^{\infty}\frac{1}{r^2-x^2}\ln\left(1+\frac{k^2}{x^2}\right)dx - (1+r^2)P\int_0^{\infty}\frac{1}{r^2-x^2}\ln\left(1+\frac{1}{k^2x^2}\right)dx.$$

By the integration of

$$\frac{1}{z^2-r^2}\ln\left(1+\frac{ik}{z}\right),$$

where $z = x+iy$, it is straightforward to verify that

$$P\int_0^{\infty}\frac{1}{r^2-x^2}\ln\left(1+\frac{k^2}{x^2}\right)dx = \frac{\pi}{r}\tan^{-1}\frac{k}{r},$$

and this result requires k to be positive. Accordingly, it is possible to obtain a closed form expression for $R(r)$, and hence

$$K(\psi) = \frac{-\pi}{\sin\psi\cos\psi}\tan^{-1}\left(\frac{2B}{(A^2-B^2)^{\frac{1}{2}}}\sin\psi\cos\psi\right).$$

It is straightforward to obtain $I(\sin^2\psi)$; and since $\sin^2\psi = u$, the solution of the integral equation is now

$$I(u) = \frac{4}{\pi}\tan^{-1}\frac{2b(u-u^2)^{\frac{1}{2}}}{\{(a-\alpha)^2-b^2\}^{\frac{1}{2}}} - \frac{2}{\pi}\tan^{-1}\frac{2b(u-u^2)^{\frac{1}{2}}}{(a^2-b^2)^{\frac{1}{2}}}.$$

We are now in the fortunate position of being able to utilize the first transformation of (12) to bring the solution to a form involving the original variable x. From (8) and (13) the current distribution can be placed in the form

$$\frac{i(x)}{i_0} = 1 - \frac{4}{\pi}\tan^{-1}\frac{(b^2-x^2)^{\frac{1}{2}}}{\{(a-\alpha)^2-b^2\}^{\frac{1}{2}}} + \frac{2}{\pi}\tan^{-1}\frac{(b^2-x^2)^{\frac{1}{2}}}{(a^2-b^2)^{\frac{1}{2}}}.$$

For convenience in further theoretical investigation and computation, we now introduce the nondimensional parameters

$$\lambda = \alpha/a, \quad \mu = b/a, \quad \omega = x/b.$$

As a consequence,

$$\frac{i(b\omega)}{i_0} = 1 - \frac{4}{\pi}\tan^{-1}\left[\frac{(1-\omega^2)\mu^2}{(1-\lambda)^2-\mu^2}\right]^{\frac{1}{2}} + \frac{2}{\pi}\tan^{-1}\left[\frac{(1-\omega^2)\mu^2}{1-\mu^2}\right]^{\frac{1}{2}}, \quad (20)$$

where $-1 \le \omega \le 1$.

4. BEHAVIOR OF THE CURRENT DISTRIBUTION

Since we have a definite analytic expression for the current distribution, we can now pursue the behavior throughout a cycle with some ease. If we regard the region of superconductivity to be given, that is, b/a or μ is known, then, during each half-cycle, α monotonically increases its value from zero to $a-b$. Accordingly, in (20) we can regard μ as being fixed and λ as varying from zero to $1-\mu$. From the description given by Fig. 1 we have

$$b < a \quad \text{or} \quad \mu < 1;\ a-\alpha > b \quad \text{or} \quad 1-\lambda > \mu. \quad (21)$$

Consequently, for ω in the range $-1 \le \omega \le 1$, the denominators of each quantity in the representation of the current distribution (20) do not become negative. Also, since $i(b\omega)$ is an even function of ω, we need only consider values of ω for which $0 \leq \omega \leq 1$.

One immediate result from (20) is that, when $\omega = \pm 1$, $i(\pm b) = i_0$, which agrees with (5). Now, by differentiation of (20) with respect to ω, we have

$$\frac{d}{d\omega}\left[\frac{i(b\omega)}{i_0}\right] = \frac{2\omega\mu(N - D\omega^2\mu^2)}{\pi(1-\omega^2)^{\frac{1}{2}}\{(1-\lambda)^2 - \omega^2\mu^2\}(1-\omega^2\mu^2)},$$

where

$$N = 2\{(1-\lambda)^2 - \mu^2\}^{\frac{1}{2}} - (1-\lambda)^2(1-\mu^2)^{\frac{1}{2}},$$
$$D = 2\{(1-\lambda)^2 - \mu^2\}^{\frac{1}{2}} - (1-\mu^2)^{\frac{1}{2}}.$$

The gradient is infinite for $\omega = \pm 1$ and is zero whenever

$$\omega = 0, \tag{22}$$

$$\omega^2 = N/\mu^2 D. \tag{23}$$

The first of these, Eq. (22), gives a minimum or a maximum turning point, accordingly as $N \gtrless 0$. For convenience, we introduce

$$A = \left[\frac{2 - 2(1 - \mu^2 + \mu^4)^{\frac{1}{2}}}{1 - \mu^2}\right]^{\frac{1}{2}}, \quad B = \tfrac{1}{2}(1 + 3\mu^2)^{\frac{1}{2}}.$$

Hence the following can be verified:

$$N > 0, \quad 1 - \lambda > A; \quad N = 0, \quad 1 - \lambda = A;$$
$$N < 0, \quad 1 - \lambda < A; \quad D > 0, \quad 1 - \lambda > B;$$
$$D = 0, \quad 1 - \lambda = B; \quad D < 0, \quad 1 - \lambda < B.$$

In the (λ, μ) plane with μ as abscissa, there are three regions: region 1, which lies above the curve $1 - \lambda = B$; region 2, which lies between the curves $1 - \lambda = B$ and $1 - \lambda = A$; and region 3, which is the very narrow region lying between $1 - \lambda = A$ and the line $1 - \lambda = \mu$. Note that the curves $1 - \lambda = A$ and $1 - \lambda = B$ do not overlap, but are tangential to each other at $\mu = 1$. In region 1, $N > 0$, $D > 0$, and $N/D > 1$. Therefore the value of ω as given by (23) is greater than unity, which is not allowed. Hence there is no turning point in this case. However, the turning point given by (22) is a minimum.

In region 2, $N > 0$, $D < 0$, and the value of ω^2 as given by (23) is negative; again there is no turning point in this case. The turning point given by (22) is a minimum.

In region 3, $N < 0$ and $D < 0$. Set $1 - \lambda = \mu + \epsilon$, where ϵ is a small positive quantity. A straightforward expansion in terms of ϵ indicates that, for values of $1 - \lambda$ close to μ, we have ω^2 less than unity; accordingly, there are now turning points given by (23). It is easy to verify that this is a minimum turning point. The turning point given by (22) is a maximum.

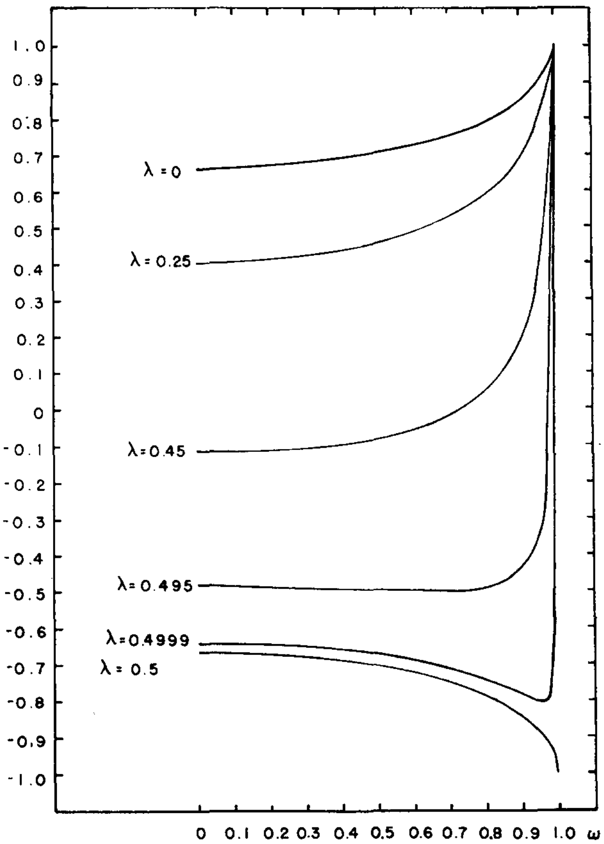

FIG. 2. Behavior of the current distribution $i(b\omega)$ for $\mu = 0.5$ and λ increasing from 0 to $1 - \mu$ per half-cycle.

The above analysis can be used to indicate the behavior of the current distribution $i(b\omega)$ through a half-cycle from $\alpha = 0$ $(\lambda = 0)$ to $\alpha = a - b$ $(\lambda = 1 - \mu)$. For when $\lambda = 0$, we have, from (20),

$$\frac{i(b\omega)}{i_0} = 1 - \frac{2}{\pi}\tan^{-1}\frac{(1-\omega^2)\mu}{(1-\mu^2)^{\frac{1}{2}}}. \tag{24}$$

As λ increases, successive curves of $i(b\omega)$ lie beneath the curve defined by (24) (see Fig. 2). At some value of λ (actually given by $N = 0$) each curve begins to be convex and to take the forms shown on the figure. When $\lambda = 1 - \mu$, Eq. (20) gives

$$\frac{i(b\omega)}{i_0} = -1 + \frac{2}{\pi}\tan^{-1}\frac{(1-\omega^2)\mu}{(1-\mu^2)^{\frac{1}{2}}},$$

which is just the exact inverse of (24). Figure 2 shows some typical curves for the current distribution for the case when μ is 0.5 and for $0 \leq \omega \leq 1$.

The importance of the existence of the closed-form solution for the above mathematical model of the superconducting strip is as follows. By means of purely physical reasoning it was possible to predict that the profile of the current distribution in the superconducting portion of the strip during the first half-cycle would be convex upward and that in the next half-cycle the profile ought to be concave downward. The above mathematical model predicts this behavior, and the analysis of the slope of the current distribution illustrates in detail how the transition during each half-cycle is effected.

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Functional Integrals for Many-Boson Systems

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We express the grand canonical partition function (GCPF) for a system of interacting bosons as a functional integral over one complex function. Our derivation is based on the so-called coherent-state representation [R. J. Glauber, Phys. Rev. **136**, 2766 (1963)]. We show how to extract the perturbative expansion of the GCPF and the various Green's functions from our functional-integral representation and we indicate the relevance of our formalism to the theory of superfluidity.

I. INTRODUCTION

The central quantity in the quantum statistics of many-boson systems is the grand canonical partition function (GCPF). It is useful to obtain a variety of mathematical expressions for this quantity, since a particular expression might prove to be particularly suitable for a discussion of a given physical phenomenon. We believe that, for a discussion of superfluidity occurring in ^{4}He at low temperature, it is useful to express the GCPF as a functional integral[1] analogous to Feynman's well-known path-integral representation of quantum-mechanical wavefunctions. The basis for this belief is discussed in Sec. V of this paper.

A functional-integral representation for the GCPF has been described some time ago by Bell[2] on the basis of a somewhat involved mathematical argument. In the present paper we present a much simpler and more physically transparent derivation of Bell's result. Our approach makes crucial use of the so-called coherent-state representation.[3] Our method has the further advantage of yielding the functional integral representation of the GCPE *directly*. In contrast, Bell's derivation yields an expression only for the ratio Z/Z_0, where Z and Z_0 are the GCPF's for the interacting and free systems, respectively. Finally, Bell's derivation relies implicitly on the assumption that the chemical potential is finite, whereas ours does not.

Our derivation of the functional integral for the GCPF is given in Sec. II. In Sec. III, we derive the perturbative expansion of the GCPF directly from its functional integral representation. In Sec. IV, we widen our outlook to include an additional interaction with an external c-number source $J(x)$. This enables us to apply formal functional-integration techniques and derive an expression for the GCPF, from which the various Green's functions of the system are easily recovered by means of functional differentiation with respect to $J(x)$. The relevance of our formalism to the treatment of superfluidity is discussed in Sec. V. Langer[4] also discusses the role of coherent states[3] in

[1] S. G. Brush, Rev. Mod. Phys. **33**, 79 (1961); I. M. Gel'fand and A. M. Yaglom, J. Math. Phys. **1**, 48 (1960).

[2] J. S. Bell, *The Many Body Problem*, E. R. Caianiello, Ed. (Academic Press Inc., New York, 1962). After the completion of this paper we discovered that S. S. Schweber, [J. Math. Phys. **3**, 831 (1962)] had used a method essentially identical to ours to derive the content of Sec. II. However, since he considers only real time and does not discuss the GCPF directly, we feel that our derivation is still useful. Schweber's paper gives references to other works on the coherent state formalism.

[3] R. J. Glauber, Phys. Rev. **136**, 2766 (1963).

[4] J. S. Langer (report of work prior to publication). We wish to thank Professor A. Ron for drawing our attention to this work.

The importance of the existence of the closed-form solution for the above mathematical model of the superconducting strip is as follows. By means of purely physical reasoning it was possible to predict that the profile of the current distribution in the superconducting portion of the strip during the first half-cycle would be convex upward and that in the next half-cycle the profile ought to be concave downward. The above mathematical model predicts this behavior, and the analysis of the slope of the current distribution illustrates in detail how the transition during each half-cycle is effected.

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Functional Integrals for Many-Boson Systems

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We express the grand canonical partition function (GCPF) for a system of interacting bosons as a functional integral over one complex function. Our derivation is based on the so-called coherent-state representation [R. J. Glauber, Phys. Rev. **136**, 2766 (1963)]. We show how to extract the perturbative expansion of the GCPF and the various Green's functions from our functional-integral representation and we indicate the relevance of our formalism to the theory of superfluidity.

I. INTRODUCTION

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A functional-integral representation for the GCPF has been described some time ago by Bell[2] on the basis of a somewhat involved mathematical argument. In the present paper we present a much simpler and more physically transparent derivation of Bell's result. Our approach makes crucial use of the so-called coherent-state representation.[3] Our method has the further advantage of yielding the functional integral representation of the GCPE *directly*. In contrast, Bell's derivation yields an expression only for the ratio Z/Z_0, where Z and Z_0 are the GCPF's for the interacting and free systems, respectively. Finally, Bell's derivation relies implicitly on the assumption that the chemical potential is finite, whereas ours does not.

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[1] S. G. Brush, Rev. Mod. Phys. **33**, 79 (1961); I. M. Gel'fand and A. M. Yaglom, J. Math. Phys. **1**, 48 (1960).

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the theory of superfluidity, but his method differs from ours, since he considers functions which depend on the space variables only.

II. FUNCTIONAL INTEGRALS

In this section we derive the functional-integral expression for a system of interacting bosons. For the sake of clarity we shall first give the derivation for a system of free bosons. All the essential points of our method appear in this simple case and the generalization to the case of interacting systems is straightforward.

We therefore seek to evaluate the grand canonical partition function[5]

$$Z = \mathrm{Tr}\, e^{-\beta \tilde{H}_0} \tag{1}$$

for a system of free bosons described by the Hamiltonian

$$H_0 = \sum_k (\epsilon_k - \mu) a_k^\dagger a_k, \tag{2}$$

where $\epsilon_k - \mu$ (with $\epsilon_k = k^2$) is the free particle energy measured from the chemical potential μ. We have set $\hbar = 2m = 1$. In Eq. (1), β is the inverse temperature in units of Boltzmann's constant and the trace involves all states and all particle numbers. The creation and destruction operators $a_k^\dagger$ and a_k obey the usual Bose–Einstein commutation rules

$$[a_k, a_{k'}] = 0,$$
$$[a_k, a_{k'}^\dagger] = \delta_{kk'}.$$

We shall assume throughout that we are dealing with spinless particles.

Let us choose as our set of states over which the trace in Eq. (1) is to be taken, the set of Glauber-coherent[3] states $|\alpha_k\rangle$. These are defined to be the eigenstates of the non-Hermitian operators a_k:

$$a_k |\alpha_k\rangle = \alpha_k |\alpha_k\rangle.$$

These states are given explicitly by

$$|\alpha_k\rangle = e^{(\alpha_k a_k{}^\dagger - \bar{\alpha}_k a_k)} |0\rangle,$$

with the bar indicating complex conjugation. The properties which are important here are the expression for the scalar product of two coherent states

$$\langle \alpha_k \mid \alpha'_{k'} \rangle = \delta_{kk'} \exp(\bar{\alpha}_k \alpha'_{k'} - \tfrac{1}{2}|\alpha_k|^2 - \tfrac{1}{2}|\alpha'_{k'}|^2) \tag{3}$$

and the completeness relation

$$\prod_k \int |\alpha_k\rangle \frac{d^2\alpha_k}{\pi} \langle \alpha_k| = 1, \tag{4}$$

where

$$d^2\alpha = d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha).$$

[5] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 5, 1417 (1960).

If we use the coherent states to represent the GCPF, we then have

$$Z_0 = \prod_k \int \frac{d^2\alpha_k}{\pi} \langle \alpha_k | e^{-\beta \tilde{\epsilon}_k a_k{}^\dagger a_k} |\alpha_k\rangle,$$

where we have

$$\tilde{\epsilon}_k = \epsilon_k - \mu.$$

Let us rewrite the matrix element that appears in the integrand in the following manner[6]:

$$\langle \alpha_k | e^{-\beta \tilde{\epsilon}_k a_k{}^\dagger a_k} |\alpha_k\rangle = \langle \alpha_k | \prod_{n=1}^N e^{-\delta\tau_n \tilde{\epsilon}_k a_k{}^\dagger a_k} |\alpha_k\rangle, \tag{5}$$

where we have set $\sum_{n=1}^N \delta\tau_n = \beta$, with $\delta\tau_n \tilde{\epsilon}_k \sim 1/N$. We shall, of course, be interested in the limit $N \to \infty$. We now omit the subscript k and write

$$\begin{aligned}
\int \frac{d^2\alpha}{\pi} \langle \alpha | \prod_{n=1}^N e^{-\delta\tau_n \tilde{\epsilon} a^\dagger a} |\alpha\rangle
&= \int \frac{d^2\alpha_0}{\pi} \langle \alpha_0 | e^{-\delta\tau_1 \tilde{\epsilon} a^\dagger a} |\alpha_1\rangle \frac{d^2\alpha_1}{\pi} \cdots \frac{d^2\alpha_m}{\pi} \\
&\times \langle \alpha_m | e^{-\delta\tau_{m+1} \tilde{\epsilon} a^\dagger a} |\alpha_{m+1}\rangle \frac{d^2\alpha_{m+1}}{\pi} \cdots \frac{d^2\alpha_{N-1}}{\pi} \\
&\times \langle \alpha_{N-1} | e^{-\delta\tau_N \tilde{\epsilon} a^\dagger a} |\alpha_0\rangle.
\end{aligned} \tag{6}$$

Here the subscript m refers to the ordering introduced in Eq. (5) and the original α_k variable has been denoted by α_0. The right-hand side of Eq. (6) can be written to the first order in $\delta\tau$ as

$$\begin{aligned}
&\int \frac{d^2\alpha_0}{\pi} \exp\{-\tfrac{1}{2}|\alpha_0|^2 + \bar{\alpha}_0\alpha_1 - \delta\tau_1 \tilde{\epsilon} \bar{\alpha}_0 \alpha_1 - \tfrac{1}{2}|\alpha_1|^2\} \cdots \\
&\times \int \frac{d^2\alpha_m}{\pi} \exp\{-\tfrac{1}{2}|\alpha_m|^2 + \bar{\alpha}_m \alpha_{m+1} \\
&\qquad - \delta\tau_{m+1} \bar{\alpha}_m \alpha_{m+1} - \tfrac{1}{2}|\alpha_{m+1}|^2\} \cdots \\
&\times \int \frac{d^2\alpha_{N-1}}{\pi} \exp\{-\tfrac{1}{2}|\alpha_{N-1}|^2 + \bar{\alpha}_{N-1}\alpha_0 \\
&\qquad - \delta\tau_N \bar{\alpha}_{N-1}\alpha_0 - \tfrac{1}{2}|\alpha_0|^2\}.
\end{aligned}$$

Here we used the formula (3) for the scalar product $\langle \alpha \mid \alpha' \rangle$. We now combine terms involving the indices (m, m) with those involving the indices $(m, m-1)$ and similarly terms involving the indices $(m, m+1)$ are grouped with the terms featuring the indices $(m+1, m+1)$. This leads to terms of the following general form:

$$\exp\{-|\alpha_m|^2 + \bar{\alpha}_{m-1}\alpha_m - \delta\tau_m \tilde{\epsilon} \bar{\alpha}_{m-1}\alpha_m\}. \tag{7}$$

Note that, for this arrangement, α_0 is considered as indexed by N except for the mixed term $\bar{\alpha}_0\alpha_1$ where α_0 is the zeroth term. The expression (7) can be written

[6] R. P. Feynman, Phys. Rev. **84**, 123 (1951).

as

$$\exp\left\{-\delta\tau_m\left[\frac{\bar{\alpha}_m-\bar{\alpha}_{m-1}}{\delta\tau_m}\alpha_m+\tilde{\epsilon}\bar{\alpha}_{m-1}\alpha_m\right]\right\}. \qquad (8)$$

We can now replace the $\bar{\alpha}_{m-1}\alpha_m$ by $\bar{\alpha}_m\alpha_m$, since the correction term is negligible when compared with the other term in the square bracket for $\tilde{\epsilon}\delta\tau_m\to 0$ which is the limit considered here. Thus, in this limit, we can now write Z in the form (restoring the k index)

$$Z_0=\prod_k\int\frac{d^2\alpha_k(\tau)}{\pi}\exp\left\{-\int_0^\beta d\tau[\partial_\tau\bar{\alpha}_k(\tau)\alpha_k(\tau)+\tilde{\epsilon}_k\bar{\alpha}_k(\tau)\alpha_k(\tau)]\right\}, \qquad (9)$$

where the integral over $\delta^2\alpha_k(\tau)$ is now interpreted as sum over all functions Re $\alpha(\tau)$ and Im $\alpha(\tau)$. If we consider Re $\alpha_k(\tau)$ and Im $\alpha_k(\tau)$ as the Fourier coefficients of Re $\psi(r,\tau)$ and Im $\psi(r,\tau)$, respectively, we can rewrite Eq. (9) in the succinct form[1]

$$Z_0=\int\delta\psi(r,\tau)\exp\left\{-\int_0^\beta d\tau\, dr[\partial_\tau\bar{\psi}(r,\tau)\psi(r,\tau)-\bar{\psi}(r,\tau)(\nabla^2+\mu)\psi(r,\tau)]\right\}$$
$$(\psi=\mathrm{Re}\,\psi+\mathrm{Im}\,\psi). \qquad (10)$$

We now wish to remark on the allowed time dependence of the $\psi(r,\tau)$. Since the time interval of interest is from 0 to β, we need consider only periodic function with a period β. A restriction on the allowed time dependence follows from the following argument. In Eq. (7) we chose to group together the $(m+1)$ term with the m term. Equally well, we could have used the grouping $(m-1,m)$. Upon following the latter prescription, the term

$$\int dr\, d\tau\partial_\tau\bar{\psi}(r,\tau)\cdot\psi(r,\tau)$$

in the action is replaced by

$$-\int dr\, d\tau\bar{\psi}(r,\tau)\cdot\partial_\tau\psi(r,\tau).$$

For consistency, these expressions must be equal. As may be ascertained by partial integration, this implies

$$\int dr\,|\psi(r,\beta)|^2=\int dr\,|\psi(r,0)|^2.$$

This requirement does not follow from the usual periodicity constraint which only imposes the equality of $\psi(r,\beta+\epsilon)$ with $\psi(r,0+\epsilon)$.

We now turn our attention to the interacting system. In this case we must evaluate [cf. Eq. (1)]

$$Z=\mathrm{Tr}\,e^{-\beta(\tilde{H}_0+V)}, \qquad (11)$$

with

$$V=\tfrac{1}{2}\sum_{k_1k_2k_3k_4}\langle k_1k_2|\,v\,|k_3k_4\rangle\, a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3}a_{k_4},$$

i.e., with a two-body interaction term which is also [cf. Eq. (2)] expressed in the second-quantized form. Again, we can rewrite the trace as

$$Z=\int\frac{d^2\alpha}{\pi}\langle\alpha|\,e^{-\beta(\tilde{H}_0+V)}\,|\alpha\rangle. \qquad (12)$$

Now, of course, we cannot split the matrix element in the integrand into a product over independent factors α_k. Nonetheless, the procedure remains formally the same as for the free-boson case—only here the differential $d^2\alpha_0/\pi$ and the unit operator

$$\int|\alpha_m\rangle\frac{d^2\alpha_m}{\pi}\langle\alpha_m|$$

stand for the product, e.g.,

$$\frac{d^2\alpha}{\pi}=\prod_k\frac{d^2\alpha_k}{\pi}.$$

With this definition we can perform the step leading to Eq. (6) as before because the Hamiltonian $\tilde{H}_0+V$ is in normal order (i.e., the creation operators appear to the left of the absorption operators whenever present). A typical exponential term in the resultant expression is

$$\exp\left\{-\tfrac{1}{2}\sum_k|\alpha_k^m|^2+\sum_k\bar{\alpha}_k^m\alpha_k^{m+1}-\delta\tau_{m+1}\left[\sum_k\bar{\alpha}_k^m\tilde{\epsilon}_k\alpha_k^{m+1}+\tfrac{1}{2}\sum_{k_1k_2k_3k_4}\bar{\alpha}_{k_1}^m\bar{\alpha}_{k_2}^m\times\langle k_1k_2|\,v\,|k_3k_4\rangle\,\alpha_{k_3}^{m+1}\alpha_{k_4}^{m+1}\right]-\tfrac{1}{2}\sum_k|\alpha_k^{m+1}|^2\right\}.$$

Finally, if we consider Re $\alpha_k(\tau)$ as a Fourier coefficient of Re $\psi(r,\tau)$ (and similarly for the imaginary part), the result can be written as a functional integral over *one* complex c-number function $\psi(r,\tau)$:

$$Z=\int\delta\psi(r,\tau)\exp\left\{\int_0^\beta d\tau\, dr\left[\bar{\psi}(r,\tau)(\nabla^2+\mu-\partial_\tau)\psi(r,\tau)-\frac{1}{2}\int dr'\bar{\psi}(r,\tau)\bar{\psi}(r',\tau)v(r,r')\psi(r',\tau)\psi(r,\tau)\right]\right\}. \qquad (13)$$

Equation (13) is our final result expressing the GCPF of a many-boson system as a functional integral analogous to Feynman's path-integral representation of a quantum-mechanical wavefunction.[7] Note that our result holds true for a wide class of interaction

[7] In fact, R. P. Feynman, [Phys. Rev. **97**, 660 (1955)] used this method to eliminate the phonons from the polaron problem. See also L. P. Kadanoff and M. Revzen, Nuovo Cimento **33**, 397 (1964).

Hamiltonians, provided that they are normal-ordered and that each term involves a finite number of creation and destruction operators. Finally, we remark that, in contrast to Bell's derivation, our approach is valid whether or not the chemical potential is finite. Thus our formalism is directly applicable to phonons for which $\mu = 0$.

III. LINKED CLUSTER EXPANSION FOR THE GCPF

In this section we outline a new method for obtaining a linked cluster expansion for $\ln Z$. This will serve as a point of contact with the standard[8] linked cluster expansions of quantum-statistical mechanics as well as providing an independent method for deriving this expansion.

Of course, the point of Eq. (13) is to provide a basis for discussing many-boson systems when the cluster (i.e., essentially perturbative) expansion is not useful. This aspect of the formula will be dealt with in Sec. V.

Since we are dealing with simple functions in Eq. (13), we can expand the exponential containing the potential V to get

$$Z = Z_0 + \int \delta\psi(r,\tau) \times \exp\left\{\int_0^\beta dr\, d\tau[\bar{\psi}(r,\tau)(\nabla^2 + \mu + \partial_\tau)\psi(r,\tau)]\right\} \times \left\{-\frac{1}{2}\int_0^\beta dr\, dr'\, d\tau \bar{\psi}(r,\tau)\bar{\psi}(r',\tau)v(r,r') \times \psi(r',\tau)\psi(r,\tau) + \frac{(-1)^2}{2!}\frac{1}{4}\int \cdots + \cdots\right\}, \quad (14)$$

with

$$Z_0 = \int \delta\psi(r,\tau) \times \exp\left\{\int_0^\beta d\tau\, dr[\bar{\psi}(r,\tau)(\nabla^2 + \mu + \partial_\tau)\psi(r,\tau)]\right\}. \quad (15)$$

At this point we remark that only terms with equal number of $\bar{\alpha}_k$ and α_k contribute for each k. This follows quite generally and will become clear when we consider below a special example. This corresponds to the fact that only terms with equal numbers of creation and absorption operators contribute in the standard linked cluster expansion.[8] We shall defer remarks pertinent to the case where five indices repeat until we consider a special example. Going back to Eq. (14), we now divide through by Z_0 and, as an explicit example, consider the first-order term:

$$Z_0^{-1}\int \prod_k \frac{d^2\alpha_k(\tau)}{\pi} \times \exp\left\{-\int_0^\beta d\tau \sum_k \bar{\alpha}_k(\tau)(k^2 - \mu - \partial_\tau)\alpha_k(\tau)\right\} \times (-)\frac{1}{2}\int_0^\beta d\tau \sum_{k_1k_2k_3k_4} \bar{\alpha}_{k_1}(\tau)\bar{\alpha}_{k_2}(\tau)\langle k_1k_2|\, v\, |k_3k_4\rangle \times \alpha_{k_3}(\tau)\alpha_{k_4}(\tau). \quad (16)$$

As was remarked above, the only contributions come from the following three cases: $k_1 = k_2 = k_3 = k_4$, $k_1 = k_3 \neq k_4 = k_2$, and $k_1 = k_4 \neq k_3 = k_2$. Since the exponential term in the numerator of Eq. (16) splits into products of different index k, all terms whose k is not present in the potential-energy term simply cancel with the corresponding term in the denominator. In fact, it is easy to see that our problem reduces to the evaluation of terms such as (omiting the index k)

$$\frac{\int \frac{\delta^2\alpha(t)}{\pi}\exp\left\{-\int_0^\beta dt\bar{\alpha}(t)(k^2 - \mu - \partial_t)\alpha(t)\right\}\bar{\alpha}(\tau_l)\alpha(\tau_n')}{\int \frac{\delta^2\alpha(t)}{\pi}\exp\left\{-\int_0^\beta dt\bar{\alpha}(t)(k^2 - \mu - \partial_t)\alpha(t)\right\}} \quad (17)$$

and terms in which more than one pair of $\bar{\alpha}(\tau)$, $\alpha(\tau')$ appear in an arbitrary order (i.e., τ can be "later" or "earlier" than τ'). [For brevity we shall consider only the term of Eq. (17) and remark on the other terms as well as the vanishing terms—i.e., those where $\bar{\alpha}$ is not paired with α.]

The denominator can be evaluated directly to give the Bose factor

$$[1 - e^{\beta(\epsilon-\mu)}]^{-1}. \quad (18)$$

The evaluation is straightforward and requires the repeated use of the identity[9]

$$\frac{1}{\pi}\int d^2\eta e^{-\gamma\bar{\eta}\eta + \lambda\eta + \nu\bar{\eta}} = \frac{1}{\gamma}e^{\lambda\nu/\gamma}, \quad (19)$$

which is valid for all complex numbers γ, η, λ, and ν with $\mathrm{Re}\,\gamma > 0$.

The numerator of Eq. (17) can also be evaluated in the same way. For the arrangement given, the integration up to τ_l is done simply; for the α_l integration it gives the factor

$$\bar{\alpha}_l e^{-(\tau_l-\delta\tau)(\epsilon-\mu)}e^{\alpha_0\bar{\alpha}_l}e^{-(\tau_l-\delta\tau)(\epsilon-\mu)}e^{-\bar{\alpha}_l\alpha_l}e^{\delta\tau(\epsilon-\mu)}e^{\bar{\alpha}_l\alpha_{l+1}}. \quad (20)$$

We now remove the $\bar{\alpha}_l$ factor (not in the exponent) by relabelling $\alpha_0 \to \alpha_0'$; then, after the α_l integration,

[8] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinsky, *Methods of Quantum Field Theory in Statistical Physics* (R. A. Silverman, translator) (Prentice-Hall, Inc., Engelwood Cliffs, N.J., 1963).

[9] B. R. Mollow and R. J. Glauber, Phys. Rev. **160**, 1076 (1967).

Eq. (20) gives the factor

$$\frac{\partial}{\partial\alpha_0'}\{e^{-\delta\tau(\epsilon-\mu)}\exp[\alpha_0'\bar{\alpha}_{l+1}e^{-\tau_l(\epsilon-\mu)}]\}. \tag{21}$$

Proceeding in the same way to the α_n term, we get the factor

$$\frac{\partial}{\partial\alpha_0'}\{e^{-(\tau_n-\tau_l-\delta\tau)(\epsilon-\mu)}e^{\alpha_0'\bar{\alpha}_n e^{-(\tau_n-\delta\tau)(\epsilon-\mu)}} \times e^{-\bar{\alpha}_n\alpha_n e^{\delta\tau(\epsilon-\mu)}}e^{\alpha_n\bar{\alpha}_{n+1}}\alpha_n\}. \tag{22}$$

Again we remove the α_n factor (which is not in the exponent) by writing it as a derivative of a tagged variable $\bar{\alpha}_{n+1}$ and then carry out the integration over the α_n variable to get

$$\frac{\partial^2}{\partial\alpha_0'\partial\bar{\alpha}_{n+1}'}\{e^{-(\tau_n-\tau_l)(\epsilon-\mu)}e^{\alpha_0'\bar{\alpha}_{n+1}'e^{-\tau_n(\epsilon-\mu)}}\}, \tag{23}$$

which leads to two terms

$$e^{-\tau_n(\epsilon-\mu)}e^{-(\tau_n-\tau_l)(\epsilon-\mu)}\left[e^{\bar{\alpha}_{n+1}\alpha_0 e^{-\tau_n(\epsilon-\mu)}} + \alpha_0\frac{\partial}{\partial\alpha_0'}e^{\alpha_0'\bar{\alpha}_{n+1}e^{-\tau_n(\epsilon-\mu)}}\right].$$

The first term in the square bracket leads to

$$e^{-\beta(\epsilon-\mu)}e^{-(\tau_n-\tau_l)(\epsilon-\mu)}(1-e^{-\beta(\epsilon-\mu)})^{-1}, \tag{24}$$

while the second term leads to

$$\frac{e^{-\beta(\epsilon-\mu)}e^{-(\tau_n-\tau_l)(\epsilon-\mu)}e^{-\beta(\epsilon-\mu)}}{(1-e^{-\beta(\epsilon-\mu)})^2}. \tag{25}$$

Adding the two terms together and dividing by Eq. (18) gives the well-known result for the free-particle Green's function[8]

$$e^{-(\tau_n-\tau_l)(\epsilon-\mu)}(1-e^{-\beta(\epsilon-\mu)})^{-1}. \tag{26}$$

Our method was such as to make reasonably clear the following:

(a) If $\bar{\alpha}$ is not paired, the integral vanishes. This follows ultimately from the symmetry of the integral.

(b) If α comes earlier than $\bar{\alpha}$, one gets an extra factor $e^{-\beta(\epsilon-\mu)}$.

(c) If more than one pair of $\bar{\alpha}$, α is present, one gets products of free-particle thermal Green's functions.[8]

We also remark that only linked terms need be considered when one takes the log (Z/Z_0). To summarize, the functional method is consistent also for the case dealt with in this paper, and the perturbation (diagrammatic) scheme[8] is derivable from it directly. (Needless to say, we do not recommend this way to get the diagrammatic expansion for Z.)

IV. EXTERNAL SOURCES AND GREEN'S FUNCTIONS

To widen the field of application of our technique, we now include an additional interaction

$$H^{\text{ext}}(\tau) = -\int(\bar{J}(\mathbf{x},\tau)\psi(\mathbf{x},\tau) + \bar{\psi}(\mathbf{x},\tau)J(\mathbf{x},\tau))\,d^3x \tag{27}$$

with an external c-number field $J(\mathbf{x},\tau)$. Physically, this amounts to probing the interacting boson system with the aid of an external source. The response of the system to this probe yields the various Green's functions of physical interest. For example, the one-particle Green's function in the presence of the external source

$$G(x,x')_J = -i\langle T\psi(x)\bar{\psi}(x')\rangle_J \tag{28}$$

[with $x=(\mathbf{x},\tau)$] is obtained by functional differentiation of the grand canonical partition function $Z[\bar{J},J]$[10]:

$$G(x,x')_J = \frac{1}{Z[\bar{J},J]}\frac{\delta}{\delta\bar{J}(x)}\frac{\delta}{\delta J(x')}Z[\bar{J},J]. \tag{29}$$

In general, an nth-order Green's function is obtained by taking the nth-order functional derivative of $Z[\bar{J},J]$.

Now the functional integral for $Z[\bar{J},J]$ is given by

$$\begin{aligned} Z[\bar{J},J] &= C\int\delta\psi(\mathbf{x},\tau)\exp\left\{-\int_0^\beta d\tau\int d\mathbf{r}\int d\mathbf{r}'\bar{\psi}(\mathbf{r},\tau)\bar{\psi}(\mathbf{r}',\tau) \times v(\mathbf{r},\mathbf{r}')\psi(\mathbf{r}',\tau)\psi(\mathbf{r},\tau)\right\} \\ &\times\exp\left\{-\int_0^\beta d\tau\int d\mathbf{r}\left[\bar{\psi}(\mathbf{r},\tau)\frac{\partial\psi(\mathbf{r},\tau)}{\partial\tau} - \bar{\psi}(\mathbf{r},\tau)(\nabla^2+\mu)\psi(\mathbf{r},\tau) - \bar{J}(\mathbf{r},\tau)\psi(\mathbf{r},\tau) - \bar{\psi}(\mathbf{r},\tau)J(\mathbf{r},\tau)\right]\right\}, \end{aligned} \tag{30}$$

which differs from (13) by the appearance of the source-dependent terms in the second exponential factor. We shall formally carry out the functional integration in (30) so as to cast $Z[\bar{J},J]$ into a form in which the functional differentiations with respect to J can be read off as a perturbative series in the interaction potential v. This will yield the linked cluster expansion for the Green's functions.

Note that we have introduced an arbitrary multiplicative constant C in the expression (30). This is due to a difference in the meaning of the symbol $\delta\psi(\mathbf{x},\tau)$ as used in (30) as compared to, say, (13). In (13), and more generally throughout Secs. II and III, the

[10] P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959); E. S. Fradkin, Zh. Eksp. Teor. Fiz. **36**, 1286 (1959) [Sov. Phys.—JETP **9**, 912 (1959)].

"volume element" in functional space $\delta\psi(\mathbf{x}, \tau)$ has been taken in the sense of the limit

$$\lim_{n\to\infty} \prod_{k} \frac{d^2\alpha_{\mathbf{k}}(\tau_0)}{\pi}\frac{d^2\alpha_{\mathbf{k}}(\tau_1)}{\pi}\cdots\frac{d^2\alpha_{\mathbf{k}}(\tau_i)}{\pi}\cdots\frac{d^2\alpha_{\mathbf{k}}(\tau_n)}{\pi}$$

as the number of points $\tau_1 \cdots \tau_i \cdots$ on the interval between $\tau_0 = 0$ and $\tau_N = \beta$ tends to infinity. On the other hand, in (30), the functional volume element is taken to be

$$\delta\psi(\mathbf{x}, \tau) = \prod_{\mathbf{k}}\prod_{k_0} \frac{d^2\alpha_{\mathbf{k}k_0}}{\pi}, \tag{31}$$

where the $\alpha_{\mathbf{k}k_0}$ are the coefficients of the expansion

$$\alpha_{\mathbf{k}}(\tau) = \sum_{k_0} \alpha_{kk_0}u_{k_0}(\tau) \tag{32}$$

of $\alpha_{\mathbf{k}}(\tau)$ in terms of some complete, orthonormal set of real functions $u_{k_0}(\tau)$ on the interval $0 \le \tau \le \beta$. This change in the definition of $\delta\psi(\mathbf{x}, \tau)$ [with the corresponding appearance of the constant C in (30) to represent the Jacobian of the functional change of variables (32)] allows us to apply to (30) the powerful functional integration techniques developed by Symanzik[11] in relativistic quantum-field theory. The constant C will be determined at the end by the requirement that $Z[\bar{J}, J]$ reduce to

$$Z_0 = [1 - e^{\beta(\epsilon-\mu)}]^{-1}$$

when $J = \bar{J} = 0$ and $v(r, r') = 0$, that is, for the noninteracting system.

To integrate (30) we start with simple formula

$$\int \delta\varphi \exp(-)\int dx\bar{\varphi}\varphi = 1 \tag{33}$$

due to Symanzik.[11] Following Symanzik, we assume that the functional integration is translation-invariant, i.e., that

$$\int \delta\varphi F[\varphi] = \int \delta\varphi F[\varphi - \chi] \tag{34}$$

for a functional $F[\varphi]$, so that, from (33), we obtain

$$\int \delta\varphi \exp\int dx[-\bar{\varphi}\varphi + \bar{\chi}\varphi + \bar{\varphi}\chi] = \exp\int dx\bar{\chi}\chi. \tag{35}$$

With the aid of (35), for an arbitrary functional $F[\bar{\varphi}, \varphi]$ we can now write

$$\begin{aligned}\int \delta\varphi F[\bar{\varphi}, \varphi] \exp\int dx[-\bar{\varphi}\varphi + \bar{\chi}\varphi + \bar{\varphi}\chi] &= \int \delta\varphi F\left[\frac{\delta}{\delta\chi}, \frac{\delta}{\delta\bar{\chi}}\right] \exp\int dx[-\bar{\varphi}\varphi + \bar{\chi}\varphi + \bar{\varphi}\chi]\\ &= F\left[\frac{\delta}{\delta\chi}, \frac{\delta}{\delta\bar{\chi}}\right] \exp\int dx\bar{\chi}\chi,\end{aligned} \tag{36}$$

[11] K. Symanzik, Z. Naturforsch. **9a**, 809 (1954). See also D. Lurié, *Particles and Fields* (John Wiley & Sons, New York, 1968).

where we have made the further assumption that the functional differentiations with respect to χ and χ^* can be removed from under the functional-integral sign.

Let us now write the expression (30) in the condensed form

$$\begin{aligned}Z = C\int \delta\varphi \exp\left\{-\int_0^\beta d\tau\int d\mathbf{r}\int d\mathbf{r}'\bar{\psi}\bar{\psi}v\psi\psi\right\}\\ \times \exp\left\{\int_0^\beta d\tau\int d\mathbf{r}[-\bar{\psi}D\psi + \bar{J}\psi + \bar{\psi}J]\right\},\end{aligned} \tag{37}$$

where

$$D = \frac{\partial}{\partial t} - \nabla^2 - \mu.$$

To be in a position to apply the formula (36), we must first change the functional variable of integration in (37) by means of the formal substitution

$$\psi \to D^{-\frac{1}{2}}\psi \tag{38}$$

to obtain

$$\begin{aligned}Z[\bar{J}, J] &= C'\int \delta\psi \exp(-)\int_0^\beta d\tau\int d\mathbf{r}\int d\mathbf{r}'\bar{\psi}D^{-\frac{1}{2}}\bar{\psi}D^{-\frac{1}{2}}vD^{\frac{1}{2}}\psi D^{\frac{1}{2}}\psi\\ &\times \exp\int_0^\beta d\tau\int d\mathbf{r}[-\bar{\psi}\psi + \bar{J}D^{-\frac{1}{2}}\psi + \bar{\psi}D^{-\frac{1}{2}}J],\end{aligned} \tag{39}$$

where we have absorbed the Jacobian of the substitution (38) into the constant C'. We now perform the functional integration with the aid of (36). This gives the result

$$\begin{aligned}Z[\bar{J}, J] = C'\exp\left\{-\int_0^\beta d\tau\int d\mathbf{r}\int d\mathbf{r}'\frac{\delta}{\delta J}\frac{\delta}{\delta J}v\frac{\delta}{\delta\bar{J}}\frac{\delta}{\delta\bar{J}}\right\}\\ \times \exp\int d\tau\int d\mathbf{r}\bar{J}D^{-1}J.\end{aligned} \tag{40}$$

Since $Z[\bar{J}, J]$ reduces to C' for $J = \bar{J} = v = 0$, we may write our final result in the form

$$\begin{aligned}\frac{Z[\bar{J}, J]}{Z_0} = \exp\Bigg\{-\int_0^\beta d\tau\int d\mathbf{r}\int d\mathbf{r}'\frac{\delta}{\delta J(\mathbf{r}, \tau)}\frac{\delta}{\delta J(\mathbf{r}', \tau)}\\ \times v(\mathbf{r}, \mathbf{r}')\frac{\delta}{\delta\bar{J}(\mathbf{r}', \tau)}\frac{\delta}{\delta\bar{J}(\mathbf{r}, \tau)}\Bigg\}\\ \times \exp\Bigg\{\int_0^\beta d\tau\int_0^\beta d\tau'\int d\mathbf{r}\\ \times \int d\mathbf{r}'J(\mathbf{r}, \tau)G_0(\mathbf{r} - \mathbf{r}', \tau - \tau')J(\mathbf{r}', \tau')\Bigg\},\end{aligned} \tag{41}$$

where we have used the identity

$$\frac{1}{\frac{\partial}{\partial t} - \nabla^2 - \mu}J(\mathbf{r}, \tau) = \int G_0(\mathbf{r} - \mathbf{r}', \tau - \tau')J(\mathbf{r}', \tau')\, d\mathbf{r}'\, d\tau', \tag{42}$$

where $G_0(r, \tau)$ is the free one-boson Green's function satisfying

$$\left(\frac{\partial}{\partial\tau} - \nabla^2 - \mu\right) G_0(\mathbf{r}, \tau) = \delta^{(3)}(\mathbf{r})\delta(\tau). \tag{43}$$

Expansion of the exponentials in (41) now yields the well-known diagrammatic expansion of the grand canonical partition function in the limit $\bar{J} = J = 0$. We have

$$\begin{aligned}
&Z[\bar{J} = J = 0] \\
&= Z_0\left(1 - \int \frac{\delta}{\delta J}\frac{\delta}{\delta J} v \frac{\delta}{\delta \bar{J}}\frac{\delta}{\delta \bar{J}} + \cdots\right) \\
&\quad \times \left(1 + \int \bar{J} G_0 J + \frac{1}{2!}\int \bar{J} G_0 J \int \bar{J} G_0 J + \cdots\right)\Big|_{\bar{J}=J=0} \\
&= -Z_0 \int d\tau \int d\mathbf{r} \int d\mathbf{r}' G_0(0, 0) v(\mathbf{r} - \mathbf{r}') G_0(0, 0) \\
&\quad - Z_0 \int d\tau \int d\mathbf{r} \int d\mathbf{r}' G_0(\mathbf{r}' - \mathbf{r}, 0) v(\mathbf{r} - \mathbf{r}') G_0(\mathbf{r} - \mathbf{r}', 0) \\
&\quad + \cdots.
\end{aligned} \tag{44}$$

In a similar way one extracts from (41) the linked cluster expansions for the various Green's functions of the system in the limit in which the external sources are switched off. Setting $\bar{J} = J = 0$ in (29), for example, one has

$$\begin{aligned}
&G(\mathbf{x}, \tau, \mathbf{r}', \tau') \\
&= \frac{1}{Z[\bar{J} = J = 0]} \frac{\delta}{\delta J(\mathbf{x}, \tau)} \frac{\delta}{\delta J(\mathbf{x}', \tau')} Z[\bar{J}, J]\Big|_{\bar{J}=J=0},
\end{aligned} \tag{45}$$

and it is straightforward to check that this yields the linked cluster expansion for the one-boson Green's function upon expansion of the exponentials in (41).

V. THEORY OF SUPERFLUIDS; LANDAU–GINZBURG EQUATION

Equation (13) would be particularly useful under the following circumstance: If a particular function $\psi'(\mathbf{r}, \tau)$ gives a large contribution to the sum Eq. (13), and if further there is no dispersion (second moment) to this function $\psi'(\mathbf{r}, \tau)$, then we might approximate the whole GCPF by the contribution from this function alone. Then it would be tempting to associate this function $\psi'(\mathbf{r}, \tau)$ with the "stiff" wavefunction of the system.[12,13] However, the problem here is immensely more complicated than this simple discussion might indicate because we are dealing with complex functions. At this point our analysis is closely related to Langer's[4] remarkable paper.

Clearly, the first step in our program is easy. Thus the functions $\psi'(\mathbf{r}, \tau)$ which give large contributions to Eq. (13) are those which give a stationary exponent, i.e., those which satisfy the Schrödinger-like equation (we have changed variables $\tau \to it$)

$$i\frac{\partial\psi(\mathbf{r}t)}{\partial t} = (\nabla^2 - \mu)\psi(\mathbf{r}, t) + \int |\psi(\mathbf{r}', t)|^2 v(\mathbf{r}, \mathbf{r}')\psi(\mathbf{r}, t). \tag{46}$$

This equation is to be interpreted as the Landau–Ginzburg[10] equation of our problem. As in Langer's case,[4] we have an exact equation that gives the maximal contribution to Z [Eq. (13)]. The approximation wherein Eq. (46) suffices—i.e., the neglect of the problem of dispersion—is tantamount to approximating the maximum of the Gibbs potential by its mean value. This is a well-known[14] feature of Landau's theory of second-order phase transitions. An important feature of the equation is its explicit dependence on μ and implicit dependence on β. Thus thermodynamic parameters enter directly into the wave equation of the system, and hence the dispersion depends on these parameters.

VI. CONCLUDING REMARKS

An exact formulation of the many-boson system in terms of a functional integration over one complex function was derived. The results were used to provide an alternative derivation of the linked cluster expansion for the interacting bose system. These results were obtained previously by Bell[2] by a completely different method. Then we argued that a merit of his formulation was that even when perturbative expansion is not useful, such as in the case presumably when superfluidity sets in, the formalism does provide a natural vehicle for the discussion. This is because the formalism lends itself to a simple derivation of the Landau–Ginsburg equation and (at least) to a formal statement of its limitation.

ACKNOWLEDGMENT

We are grateful to Professor C. Kuper for several helpful conversations on the contents of this work.

[12] F. London, *Superfluids Vol.1.II* (John Wiley & Sons, New York, 1954).

[13] P. G. de Gennes, *Superconductivity of Metals and Alloys* (W. A. Benjamin, Inc., New York, 1966).

[14] L. P. Kadanoff *et al.*, Rev. Mod. Phys. **30**, 395 (1967).

Generalization of the "Schwarzschild Surface" to Arbitrary Static and Stationary Metrics*

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A generalization of the $r = 2m$ "Schwarzschild surface" is defined for static metrics which are not necessarily spherically symmetric. This surface exhibits simultaneously the properties of being a "one-way membrane" for causal propagation and of being a surface of infinite red shift. The necessary and sufficient condition that these two phenomena take place on the same surface in an arbitrary stationary metric is also obtained. The distinctions between the static and stationary cases are shown to be essential by examples from the Kerr metric.

I. INTRODUCTORY REMARKS

The "Schwarzschild surface" at $r = 2m$ in the Schwarzschild exterior metric displays several interesting properties which are well known. Two of them are: (1) with reference to static sources and observers, infinite red shift takes place at the surface; (2) the surface is a null surface, so that it acts locally as a one-way membrane[1] (all 4-velocities in the future light cone cross the surface in the same direction). The question arises whether the two phenomena are interrelated and whether it is possible to characterize, in an arbitrary metric, surfaces exhibiting one or both of the above properties. This question can be answered in case of static and stationary metrics; the timelike Killing vector admitted by these metrics makes it possible to analyze the problem in a completely coordinate independent manner.

In Sec. II, the well-known red-shift formula is derived, for the sake of completeness, explicitly in terms of the Killing vector, which shows that infinite red shift results at the surface Σ_0 on which the Killing vector becomes null. It will be proved that, in an arbitrary static metric, Σ_0 is necessarily a null surface which means that Σ_0 is both an infinite red-shift surface and a one-way membrane as in the case Schwarzschild exterior metric. Similarly, in the case of stationary metrics, the necessary and sufficient condition that the surface on which the Killing vector becomes null be itself a null surface is obtained. In Sec. III, we study the Kerr metric as a specific example of these considerations. It is seen how the two "Schwarzschild" properties of infinite red shift and of "one-way" causality will typically *not* coincide, in contrast to the Schwarzschild and other static metrics where they occur at the same surface.

These questions have also been studied independently by Carter,[2] and it is hoped that the present paper can serve as an elementary, although incomplete, introduction to his more general and broadly based study.

II. GENERALIZED "SCHWARZSCHILD SURFACES"

A static metric admits a timelike Killing vector field ξ^a (Latin indices run from 0 to 3), the trajectories of which form a normal congruence.[3] Hence, the Killing vector satisfies the Killing equation

$$\xi_{a;b} + \xi_{b;a} = 0 \tag{1}$$

and the condition for a normal congruence[4] $\xi_{[a}\xi_{b;c]} = 0$. From the antisymmetry of $\xi_{a;b}$, this last equation reduces to

$$\xi_a\xi_{b;c} + \xi_b\xi_{c;a} + \xi_c\xi_{a;b} = 0. \tag{2}$$

We can define "static" observers or sources to be those with 4-velocities which satisfy[3]

$$u^a = (-\xi_b\xi^b)^{-\frac{1}{2}}\xi^a. \tag{3}$$

The frequency ν that an observer of 4-velocity u^a assigns to a light ray with geodesic tangent k^a is $\nu = -u^a k_a$ so the general red-shift formula is given by

$$\nu_o/\nu_s = (k_a u^a)_o/(k_a u^a)_s, \tag{4}$$

where the subscripts s and o stand for the source and the observer. For "static" sources and observers Eq. (3) reduces this to

$$\nu_o/\nu_s = (-\xi_a\xi^a)^{\frac{1}{2}}_s/(-\xi_a\xi^a)^{\frac{1}{2}}_o, \tag{5}$$

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[1] D. Finkelstein, Phys. Rev. **110**, 965 (1958).

[2] B. Carter (report of work prior to publication).
[3] J. Ehlers in *Gravitation: An Introduction to Current Research*, Louis Witten, Ed. (John Wiley & Sons, Inc., New York, 1962).
[4] Square brackets denote antisymmetrization:

$$\xi_{[a}\xi_{b;c]} = \tfrac{1}{6}[\xi_a\xi_{b;c} + \xi_b\xi_{c;a} + \xi_c\xi_{a;b} - \xi_b\xi_{a;c} - \xi_a\xi_{c;b} - \xi_c\xi_{b;a}].$$

We use a metric with signature $- + + +$.

where use has been made of the fact that, along a null geodesic, the product $\xi_a k^a$ is a constant.[5]

An analog of the "Schwarzschild surface" is given by $\Sigma_0: \xi_a \xi^a = 0$, for which Eq. (5) yields infinite red shift. (Since no timelike u^a is actually defined on Σ_0, this is meant as a limit; i.e., near Σ_0 red shifts may be arbitrarily large.) This condition is important, as we shall see in Sec. III when we study the Kerr metric.

Next, consider the family of surfaces Σ given by

$$\xi_a \xi^a = \text{const.} \tag{6}$$

In order to be sure that Σ, defined in this way, is a regular 3-dimensional hypersurface in 4-space, we shall assume that the gradient of $\xi_a \xi^a$ does not vanish on Σ. Then the vector

$$n_a = \tfrac{1}{2}(\xi_b \xi^b)_{;a} = \xi_{b;a}\xi^b \tag{7}$$

is nonzero and is normal to Σ. We readily verify that ξ^a lies in Σ since it is orthogonal to n_a (use the antisymmetry of $\xi_{a;b}$):

$$n_a \xi^a = \xi_{b;a}\xi^b \xi^a = 0.$$

Now compute the length of the normal vector:

$$n_b n^b = (\xi_{a;b}\xi^a)(\xi^{c;b}\xi_c) = (\xi_{a;b}\xi_c)(\xi^{c;b}\xi^a).$$

By use of Eq. (2), it can be shown that

$$n_b n^b = \tfrac{1}{2}\xi_a \xi^a (\xi_{b;c}\xi^{b;c}). \tag{8}$$

We see that $n_b n^b$, therefore, vanishes when $\xi_a \xi^a$ does, so the surface Σ_0 where $\xi_a \xi^a = 0$ is a *null* surface.

Now all null surfaces are "one-way membranes" for causal effects, but this is usually uninteresting. For example, the surface $z = t$ in Minkowski space is null (we have $c = 1$) and "one way" in the sense that future-directed timelike curves can only cross this surface in the direction of decreasing z; to cross it in the sense of increasing z would mean travelling faster than light. Every null surface such as Σ_0 has local properties similar to this standard example; namely, it contains at each point *exactly one* null direction (which is also the normal to the surface) but no time vector. The future null cone therefore lies entirely on one side of the null surface, so that all future-directed timelike directions cross the null surface in the same sense. What is remarkable about the null surface Σ_0, where $\xi_a \xi^a = 0$, is that it *does not extend to spatial infinity* (where $\xi_a \xi^a = -1$), so the light rays (null geodesics) it contains neither come from nor escape to infinity. In fact, these light rays "stand still" in the sense that their tangent k^a is parallel to ξ^a, the Killing vector which defines the idea of "static," "at rest," or "time-independent" in this metric. To see this, note that since n_a and ξ_a are both null vectors lying in Σ_0, they must be proportional there with $n_a = f\xi_a$. But then Eq. (7) reads $\xi^a{}_{;b}\xi^b = -f\xi^a$, which shows that ξ^a is parallel to a geodesic tangent k^a.

The situation in stationary metrics is considerably different from that in static metrics due to the fact that the trajectories of the timelike Killing vector field ξ^a no longer form a normal congruence, but, on the other hand, contain rotational terms.

As in the case of static metrics, we now define "stationary" observers or sources to be those with 4-velocities which satisfy

$$u^a = e^{-\psi}\xi^a, \quad u^a u_a = -1.$$

The covariant derivative of the 4-velocity has the expansion[6]

$$u_{a;b} = -\dot{u}_a u_b - (-g)^{\frac{1}{2}}\epsilon_{abrs}a^r u^s, \tag{9}$$

where

$$\dot{u}_a = u_{a;b}u^b = \psi_{,a}$$

and

$$a^r = \tfrac{1}{2}(-g)^{-\frac{1}{2}}\epsilon^{rspq}u_s u_{p;q}.$$

Here a^r is the rotation vector of the 4-velocity u^a. As a result, Eq. (2) is modified into the form

$$\xi_{a;b}\xi_c + \xi_{b;c}\xi_a + \xi_{c;a}\xi_b = -(-g)^{\frac{1}{2}}a^r\xi^s[\epsilon_{abrs}\xi_c + \epsilon_{bcrs}\xi_a + \epsilon_{cars}\xi_b]. \tag{10}$$

Nevertheless, Eq. (5) still holds for the 4-velocities u^a which now define "stationary" observers and sources. The surface on which ξ^a becomes null is once again an infinite red-shift surface with respect to these sources and observers. On the other hand, a straightforward calculation using Eq. (10) leads to the result

$$n^b n_b = \tfrac{1}{2}[\xi_a \xi^a (\xi_{b;c}\xi^{b;c}) - \omega_r \omega^r],$$

where $\omega^r = (-g)^{-\frac{1}{2}}\epsilon^{rspq}\xi_s \xi_{p;q}$, so ω^r is the rotation vector associated with the Killing vector trajectories. Hence we have the theorem that the surface on which the Killing vector becomes null will itself be a null surface if and only if the rotation vector of the Killing vector field also becomes null on it. Only under this condition will the infinite red-shift surface act as a one-way membrane also.

We may note in passing that, in both static and stationary metrics, the two vector fields ξ^a and n^a yield a natural generalization of the r–t two-surfaces of the Schwarzschild metric, since the tangent 2-planes they define are surface-forming according to

[5] This is shown by a well-known computation

$$(\xi_a k^a)_{;b}k^b = \xi_{a;b}k^a k^b + \xi_a k^a{}_{;b}k^b = 0,$$

using the (Killing) antisymmetry of $\xi_{a;b}$ and the geodesic equation for k^a.

[6] G. Salzman and A. H. Taub, Phys. Rev. 95, 1959 (1954).

Frobenius' theorem,[7] for the Lie derivative of n^a with respect to ξ^a is $\mathfrak{L}_\xi(n^a) = 0$. In fact, if we think of $\xi_a \xi^a = -e^{2\psi}$ as defining a generalization of the Newtonian gravitational potential ψ [as is reasonable in view of the red-shift formula of Eq. (5)], then n^a is a vector in the direction of the field lines (along the potential gradient), and these r–t two-surfaces are swept out by the field lines (trajectories of n^a) under the time translation generated by ξ^a. The same Newtonian imagery helps again if we ask whether any "radial geodesics" can be found, that is, geodesics confined to a ξ^a–n^a two-surface. That the answer is usually "no" one can verify by calculation, or understand by considering that even in Newtonian mechanics particles move along a single "line of force" with their velocity and acceleration parallel only under conditions of exceptional symmetry, as in the case of a particle moving along an axis of symmetry.

III. KERR METRIC

The Kerr metric[8] has the form

$$g_{ab} = \eta_{ab} + 2Hk_a k_b, \tag{11}$$

where η_{ab} is the metric of Minkowski space, k_a a null vector field, and H a scalar field. In explicit form, the line element is given by

$$\begin{aligned} ds^2 = {} & dr^2 + 2a \sin^2\theta \, dr \, d\phi \\ & + (r^2 + a^2) \sin^2\theta \, d\phi^2 + \chi \, d\theta^2 \\ & - dt^2 + (2mr/\chi)(dr + a \sin^2\theta \, d\phi + dt)^2, \end{aligned}$$

where m and a can, respectively, be identified with the mass and the angular momentum per unit mass of the source, and where

$$\chi(r, \theta) = r^2 + a^2 \cos^2\theta.$$

Since the metric components are independent of the time coordinate t, the timelike Killing vector will be

$$\boldsymbol{\xi}_t \equiv (\xi^t, \xi^r, \xi^\theta, \xi^\phi) = (1, 0, 0, 0)$$

(the subscript t has been used to distinguish the timelike Killing vector from the other Killing vectors we shall encounter) and

$$(\boldsymbol{\xi}_t)^2 = g_{00} = -(r^2 - 2mr + a^2 \cos^2\theta)/\chi.$$

[7] See Theorem 5.1 in S. Sternberg, *Lectures on Differential Geometry* (Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964) or Theorem 8-4 in L. Auslander and R. E. MacKenzie, *Introduction to Differentiable Manifolds* (McGraw-Hill Book Company, Inc., New York, 1963). Sufficient for our purposes here is also a comment on p. 105 in J. A. Schouten, *Ricci Calculus* (Springer-Verlag, Berlin, 1954), 2nd ed.

[8] Some of the basic equations in this section have been taken from the preprint "Maximal Analytic Extension of the Kerr Metric" by R. H. Boyer and R. W. Lindquist. See also R. H. Boyer and T. G. Price, Proc. Camb. Phil. Soc. **61**, 531 (1965).

Consequently, the Killing vector becomes null on surfaces where

$$r^2 - 2mr + a^2 \cos^2\theta = 0. \tag{12}$$

This equation has the solutions

$$\begin{aligned} r_0 &= m + (m^2 - a^2 \cos^2\theta)^{\frac{1}{2}}, \\ r_i &= m - (m^2 - a^2 \cos^2\theta)^{\frac{1}{2}}. \end{aligned}$$

Outside the outer surface r_0, we can have stationary sources and observers with 4-velocities following the Killing-vector trajectories and for these and only these infinite red shift occurs at the surface r_0. On the other hand, a surface $f(r, \theta) = \text{const}$ will be null only if the following equation is satisfied:

$$(r^2 - 2mr + a^2)\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial \theta}\right)^2 = 0. \tag{13}$$

The surface given by Eq. (12) does not satisfy this condition and therefore the surface r_0 is nonnull and does not act as a one-way membrane. Here is an instance of the two phenomena of infinite red shift and one-way membrane not taking place at the same surface. However, as Boyer and Price[8] have pointed out, we do have stationary null surfaces given by

$$r^2 - 2mr + a^2 = 0 \tag{14}$$

or

$$r_+ = m + (m^2 - a^2)^{\frac{1}{2}}, \quad r_- = m - (m^2 - a^2)^{\frac{1}{2}}, \tag{15}$$

which are physically significant for $a^2 \leq m^2$.

The null vector field k^a inherent in the Kerr metric is given by

$$\mathbf{k} \equiv (k^t, k^r, k^\theta, k^\phi) = (1, -1, 0, 0).$$

This shows that the future-null cone points inwards at the two null surfaces. Particles and light can only enter, but not leave, these surfaces. (These null surfaces r_+ and r_- and the infinite red-shift surface $r_0(\theta)$ are sketched in Fig. 1.) Concentrating on the outer surface r_+, we find that we cannot think of stationary observers along $\boldsymbol{\xi}_t$ on this surface, since the surface r_+ lies within the surface r_0, the two touching each other at $\theta = 0, \pi$, and $\boldsymbol{\xi}_t$ is spacelike in the intermediate region between the two surfaces. Even otherwise, r_+ is not an infinite red-shift surface as $\boldsymbol{\xi}_t$ does not become null on it. Nevertheless, we can find a set of "pseudostationary" observers and sources for whom infinite red shift still occurs on r_+. This is done as follows.

In addition to $\boldsymbol{\xi}_t$, the Kerr metric admits $\boldsymbol{\xi}_\phi$, the Killing vector associated with rotation about the axis. We form a "mixed Killing vector" $\boldsymbol{\xi}_\alpha$, defined

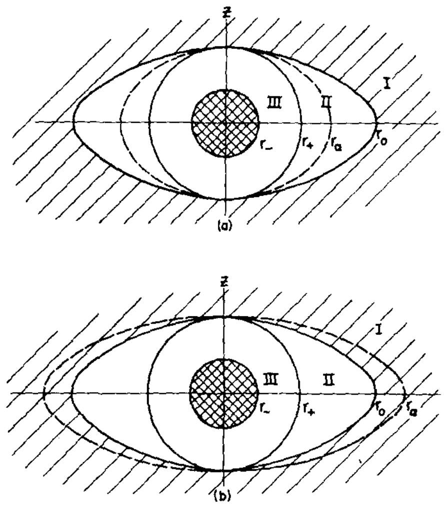

FIG. 1. Some surfaces of interest in the Kerr metric. The surfaces r_- and r_+ are null surfaces. The timelike Killing vector ξ_t becomes null on the surface r_0 which gives infinite red shift for stationary sources. The "mixed" Killing vector ξ_α becomes null on r_+ and r_α, the latter surface being nonnull. In the hatched region (region I), ξ_t is timelike so this region admits stationary observers and sources, while ξ_t is spacelike in regions II and III. The vector ξ_α is spacelike outside r_α and timelike in the region between r_+ and r_α. The cross-hatched region enclosed by the surface r_- contains the inner surface r_1 (not shown), on which ξ_t again becomes null, and the singularity $r = 0$. The future light cone points inwards at r_+ (and at r_-) so that particles and light rays can only enter, but not leave, r_+ (respectively, r_-). The diagramshave been drawn: (a) for high values of the parameter a, i.e., for a in the neighborhood of m: the surface r_α lies within r_0 and there is no common region in which both ξ_t and ξ_α are timelike. When a is equal to m, the surface r_α coincides with r_+. (b) For low nonzero values of a: surface r_0 lies within the surface r_α and in the region between these two surfaces both ξ_t and ξ_α are timelike. For $a = 0$, the Kerr metric goes over to the Schwarzschild metric, the surface r_0 and r_+ coalesce into the Schwarzschild sphere $r = 2m$, r_- collapses into the origin $r = 0$, and r_α ceases to exist.

by

$$\xi_\alpha = p \sin \alpha \xi_t + \cos \alpha \xi_\phi = (p \sin \alpha, 0, 0, \cos \alpha),$$

where p has the dimension of length and α is the mixing parameter. We wish to determine α and p for which ξ_α becomes null on the surface r_+. We compute

$$\begin{aligned}(\xi_\alpha)^2\chi = &-(r^2 - 2mr + a^2 \cos^2 \theta)p^2 \sin^2 \alpha \\ &+ [(r^2 + a^2)(r^2 + a^2 \cos^2 \theta) \sin^2 \theta \\ &+ 2mr\, a^2 \sin^4 \theta] \cos^2 \alpha \\ &+ 4mrap \sin^2 \theta \sin \alpha \cos \alpha. \qquad (16)\end{aligned}$$

Substituting $r = r_+ = m + (m^2 - a^2)^{\frac{1}{2}}$, we readily obtain

$$\sin^2 \alpha = [1 + (ap/2mr_+)^2]^{-1}.$$

A choice would be

$$ap = 2mr_+, \quad \sin \alpha = 1/\sqrt{2}, \quad \cos \alpha = -1/\sqrt{2}.$$

With these parameters,

$$\xi_\alpha = \frac{1}{\sqrt{2}}\left(\frac{2mr_+}{a}\xi_t - \xi_\phi\right). \qquad (17)$$

A second sheet $r = r_\alpha(\theta)$ on which ξ_α becomes null (but which is not a null surface) can be found from the other roots of the equation

$$(\xi_\alpha)^2\chi = 0,$$

where p and α have the above values. It is cumbersome to obtain explicit solutions to this equation in terms of θ. However, the solutions for r at $\theta = 0, \pi$, and $\pi/2$ can easily be worked out and provide enough information. At $\theta = 0, \pi$, this equation has the only possible root $r = r_+$, whereas at $\theta = \pi/2$, it admits two acceptable roots:

$$r_1 = r_+ \quad \text{and} \quad r_2 = \frac{r_+}{2}\left[\left(1 + \frac{8mr_+}{a^2}\right)^{\frac{1}{2}} - 1\right].$$

For $m > a$, $r_2 > r_1$ so that the null surface $r = r_+$ lies inside the second sheet and the two touch each other at $\theta = 0, \pi$; ξ_α becomes spacelike at infinity and hence is timelike in the region between the above two surfaces. Therefore, we can define the "pseudostationary" sources and observers in this region with 4-velocities along ξ_α. Although these have no global significance, since such observers and sources cannot be found at infinity, it is still worthwhile noting that, for these, infinite red shift does occur on the null surface r_+. This surface, rather than the surface r_0, resembles the Schwarzschild surface in that it is a one-way membrane (it exhibits infinite red shift with respect to the "pseudostationary" observers as well). This choice is borne out also by an analysis of null geodesics in the equatorial plane,[8] which shows that light signals can be sent to spatial infinity (from sources moving along timelike curves) from every point in the region between the surfaces r_0 and r_+, whereas no signal can escape from within the surface r_+.

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Interaction of Particlelike Solutions in a Classical Nonlinear Model Field Theory

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The Lagrangian for the interaction of two particles is evaluated in a model nonlinear scalar field theory. A trial function suggested by a Rayleigh–Ritz procedure is used and an exact calculation is performed. The terms in the interaction are expressed as complete elliptic integrals.

The problem of finding particlelike solutions to nonlinear model field theories has been studied recently by Rosen[1,2] and by Derrick and Kay-Kong.[3] Rosen has shown that a nonlinear model scalar field theory based on the Lagrangian density

$$\mathcal{L} = \dot{\theta}^2 - (\nabla\theta)^2 - f(\theta), \quad \theta \equiv \theta(\mathbf{r}, t), \tag{1}$$

has a singularity-free, static, spherically symmetric solution for the case $f(\theta) = -g\theta^6$. This solution to the derived scalar wave equation

$$-\ddot{\theta} + \nabla^2\theta + 3g\theta^5 = 0 \tag{2}$$

is

$$\theta = \theta_0(\mathbf{r}) = \frac{z}{[z^4 g + (\mathbf{r})^2]^{\frac{1}{2}}}. \tag{3}$$

Here g is a positive coupling constant and z is a "size parameter" for this metastable particlelike solution.

To investigate the "interaction" of two such particlelike solutions, we assume an approximate solution of the form[2,3]

$$\theta = \theta_0(1) + \theta_0(2),$$
$$\theta_0(i) \equiv \theta_0(\mathbf{r} - \boldsymbol{\xi}_i) = \frac{z_i}{[z_i^4 g + (\mathbf{r} - \boldsymbol{\xi}_i)^2]^{\frac{1}{2}}}, \tag{4}$$

where the "coordinate" $\boldsymbol{\xi}_i = \boldsymbol{\xi}_i(t)$ will describe the motion of the ith particle. Here $\theta_0[\mathbf{r} - \boldsymbol{\xi}_i(t)]$ are exact solutions to the equation

$$\nabla^2\theta_0 + 3g\theta_0^5 = 0. \tag{5}$$

The Lagrangian density then becomes

$$\mathcal{L} = (\dot{\theta}_0(1) + \dot{\theta}_0(2))^2 - (\nabla\theta_0(1) + \nabla\theta_0(2))^2 + g(\theta_0(1) + \theta_0(2))^6. \tag{6}$$

We can write

$$L = \int \mathcal{L} d^3r = T - V = \sum_{i,j} T_{ij} - \sum_{i,j} V_{ij} + V_g, \tag{7}$$

where

$$T_{ij} = \int \dot{\theta}_0(i)\dot{\theta}_0(j) d^3r,$$
$$V_{ij} = \int \nabla\theta_0(i) \cdot \nabla\theta_0(j) d^3r, \tag{8}$$
$$V_g = g \int (\theta_0(1) + \theta_0(2))^6 d^3r.$$

Integrating by parts in the expression for V_{ij}, after discarding the surface term and remembering that $\theta_0(j)$ obeys (5), we have

$$V_{ij} = -\int \theta_0(i)\nabla^2\theta_0(j) d^3r = 3g \int \theta_0(i)\theta_0^5(j) d^3r.$$

Noting that

$$\int \theta_0^6(1) d^3r = \int \theta_0^6(2) d^3r \tag{9}$$

and

$$\int \theta_0(1)\theta_0^5(2) d^3r = \int \theta_0^5(1)\theta_0(2) d^3r,$$

the velocity-independent potential terms may be combined to give

$$V = -g \int [-4\theta_0^6 + 6\theta_0^5(1)\theta_0(2) + 15(\theta_0^4(1)\theta_0^2(2) + \theta_0^2(1)\theta_0^4(2)) + 20\theta_0^3(1)\theta_0^3(2)] d^3r. \tag{10}$$

The evaluation of integrals of the type $\int \theta_0^a(1)\theta_0^b(2) d^3r$ is discussed in the Appendix.

To evaluate the integrals appearing in T, we use

$$\dot{\theta}_0(i) = \dot{\boldsymbol{\xi}}_i \cdot \nabla\theta_0(i).$$

Then,

$$T_{11} = \int \dot{\theta}_0^2(1) d^3r = \int (\dot{\boldsymbol{\xi}}_1 \cdot \nabla\theta_0(1))^2 d^3r$$
$$= \sum_{\alpha,\beta} \dot{\xi}_{1\alpha}\dot{\xi}_{2\beta} \int [\nabla_\alpha\theta_0(1)][\nabla_\beta\theta_0(1)] d^3r$$
$$= \sum_{\alpha,\beta} \dot{\xi}_{1\alpha}\dot{\xi}_{1\beta}\delta_{\beta\alpha} \times \tfrac{1}{3} \int [\nabla\theta_0(1)]^2 d^3r = \tfrac{1}{3}\dot{\boldsymbol{\xi}}_1^2 V_{11}, \tag{11}$$

[1] G. Rosen, J. Math. Phys. 6, 1269 (1965).
[2] G. Rosen, J. Math. Phys. 8, 573 (1967).
[3] G. H. Derrick and W. Kay-Kong, J. Math. Phys. 9, 232 (1968).

$$\begin{aligned}T_{12} = T_{21} &= \int \dot\theta_0(1)\dot\theta_0(2)d^3r\\ &= \int [\dot{\boldsymbol{\xi}}_1\cdot\nabla\theta_0(1)][\dot{\boldsymbol{\xi}}_2\cdot\nabla\theta_0(2)]d^3r\\ &= \sum_{\alpha,\beta} \dot\xi_{1\alpha}\dot\xi_{2\beta}\int \nabla_\alpha\theta_0(1)\nabla_\beta\theta_0(2)d^3r\\ &= -\sum_{\alpha,\beta} \dot\xi_{1\alpha}\dot\xi_{2\beta}\int \theta_0(1)\nabla_\alpha\nabla_\beta\theta_0(2)d^3r. \qquad (12)\end{aligned}$$

The further evaluation of this last integral is discussed in the Appendix.

The total field energy associated with the single-particle solution θ_0 is

$$\int [(\nabla\theta_0)^2 - g\theta_0^6]\,d^3r = 2g\int \theta_0^6\,d^3r = \frac{\pi^2}{2g^{\frac{1}{2}}} \equiv m_0. \qquad (13)$$

After performing all integrations and separating out the center-of-mass motion, we find the Lagrangian for the relative motion of two particles with size parameters which are equal in magnitude ($z_1 = \pm z_2 \equiv z$) to be

$$L = \frac{m_0}{4}(\alpha^\pm \dot R^2 + \beta^\pm R^2\dot\phi^2) - m_0\gamma^\pm, \qquad (14)$$

where

$$\alpha^\pm = \alpha^\pm(R) = 1 \mp \frac{32}{\pi}\frac{D(k)}{(R^2+4)^{\frac{3}{2}}}, \qquad (14')$$

$$\beta^\pm = \beta^\pm(R) = 1 \mp \frac{8}{\pi}\frac{B(k)}{(R^2+4)^{\frac{1}{2}}}, \qquad (14'')$$

$$\gamma^\pm = \gamma^\pm(R) = \left[2 \mp \frac{16}{\pi}\frac{B(k)}{(R^2+4)^{\frac{1}{2}}} - \frac{60}{R^2+4} \mp \frac{352}{\pi}\frac{D(k)}{(R^2+4)^{\frac{3}{2}}}\right], \qquad (14''')$$

$$R \equiv |\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1|, \quad k \equiv R/(R^2+4)^{\frac{1}{2}},$$

and $D(k)$ and $B(k)$ are the complete elliptic integrals[4]

$$D(k) = \int_0^{\pi/2} \frac{\sin^2\zeta\, d\zeta}{(1-k^2\sin^2\zeta)^{\frac{1}{2}}},$$

$$B(k) = \int_0^{\pi/2} \frac{\cos^2\zeta\, d\zeta}{(1-k^2\sin^2\zeta)^{\frac{1}{2}}}. \qquad (15)$$

Length units are chosen such that $z^2g^{\frac{1}{2}} = 1$. The velocity-dependent parts of the interaction have been incorporated into the kinetic energy terms, giving rise to reduced (enhanced) effective radial and transverse masses $m_0\alpha^+(R)[m_0\alpha^-(R)]$ and $m_0\beta^+(R)[m_0\beta^-(R)]$ for like (unlike) particles. The functions $\alpha^\pm(R)$ and $\beta^\pm(R)$ are plotted in Fig. 1.

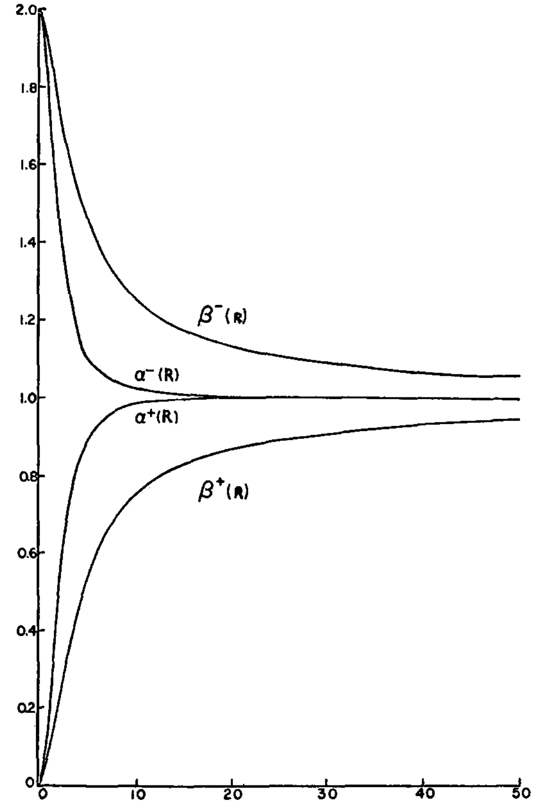

FIG. 1. Plot of the functions $\alpha^\pm(R)$, $\beta^\pm(R)$ (dimensionless).

As first quadratures of the Euler–Lagrange equations derived from (14), we have the conserved angular momentum

$$l = \frac{m_0}{2}\beta R^2\dot\phi = \text{const} \qquad (16)$$

and the conserved energy

$$E = \frac{m_0}{4}\alpha\dot R^2 + \frac{l^2}{m_0\beta r^2} + m_0\gamma = \text{const.} \qquad (17)$$

The leading terms in the series expansions of $B(k)$ and $D(k)$ valid for large R, i.e., $k \approx 1$, are

$$B(k) = 1 - \frac{(2\Lambda - 3)}{R^2+4}, \quad D(k) = \Lambda - 1 + \frac{3\Lambda - 4}{R^2+4},$$

where

$$\Lambda = \ln([2(R^2+4)]^{\frac{1}{2}}).$$

Asymptotically for large R, we have

$$\frac{B(k)}{(R^2+4)^{\frac{1}{2}}} \to \frac{1}{R}, \quad \frac{D(k)}{(R^2+4)^{\frac{3}{2}}} \to \frac{\ln R}{R^3},$$

checking the previous approximate results.[2,3]

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[4] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, Inc., New York, 1945).

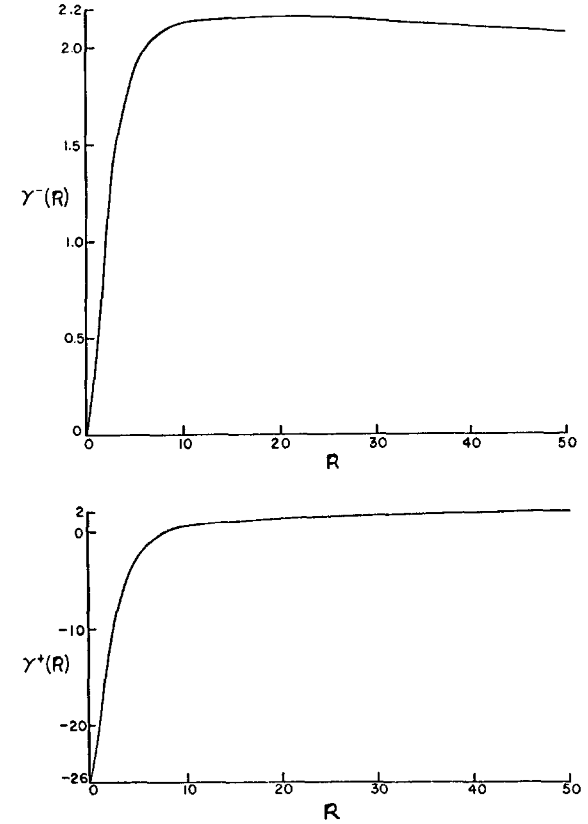

FIG. 2. Plot of the functions $\gamma\pm(R)$.

From Fig. 2 we see that the potential $V(R) = m_0\gamma^{\pm}(R)$ is attractive for like particles at all R; for unlike particles it is repulsive for large R, but becomes attractive for $R \leqslant 22$. We may assume that our trial function is close to the true solution if the overlap of the particles is small, i.e., if $|\boldsymbol{\xi}_1(t) - \boldsymbol{\xi}_2(t)| \gg z^2 g^{\frac{1}{2}}$ or $R \gg 1$.

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The author wishes to thank Professor G. Rosen for his suggestion of this problem and for his continued interest in it.

APPENDIX

The integrals occurring in $V(R)$, the velocity-independent part of the interaction potential, are of the form

$$\int \theta_0^a(1)\theta_0^b(2)d^3r = \int \frac{z_1^a z_2^b d^3r}{[z_1^4 g + (\mathbf{r} - \boldsymbol{\xi}_1)^2]^{a/2}[z_2^4 g + (\mathbf{r} - \boldsymbol{\xi}_2)^2]^{b/2}}. \quad \text{(A1)}$$

Introducing new variables

$$\mathbf{x} \equiv \frac{\mathbf{r} - \boldsymbol{\xi}_1}{g^{\frac{1}{2}}|z_1 z_2|}, \quad \mathbf{R} = \frac{\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1}{g^{\frac{1}{2}}|z_1 z_2|}, \quad Q = z_1/z_2, \quad \text{(A2)}$$

the integral (A1) becomes

$$\frac{z_1^{3-b} z_2^{3-a}}{g^{(a+b-3)/2}} \int \frac{d^3x}{(Q^{-2} + \mathbf{x}^2)^{a/2}[Q^2 + (\mathbf{x} - \mathbf{R})^2]^{b/2}}, \quad \text{(A3)}$$

which, after performing the angular integration, is equal to

$$2\pi \frac{z_1^{3-b} z_2^{3-a}}{g^{(a+b-3)/2}(b-2)R} \times \int_{-\infty}^{\infty} \frac{x\,dx}{(Q^{-2} + x^2)^{a/2}[Q^2 + (x - R)^2]^{b/2-1}}, \quad \text{(A4)}$$

if $b \neq 2$. For the case $a = 6$, $b = 0$, Eq. (A3) can be readily evaluated to give

$$\int \theta_0^6(i) d^3r = \pi^2/4g^{\frac{3}{2}}.$$

For $a = 2$, $b = 4$, Eq. (A4) can be evaluated by contour integration yielding

$$\int \theta_0^4(1)\theta_0^2(2)d^3r = \int \theta_0^2(1)\theta_0^4(2)d^3r = \frac{\pi^2}{g^{\frac{3}{2}}} \frac{1}{[R^2 + (Q + Q^{-1})^2]}. \quad \text{(A5)}$$

The cases where a and b are odd lead to elliptic integrals. For the special case of size parameters which are equal in magnitude ($Q = \pm 1$), the integrals can be expressed in terms of complete elliptic integrals. In the general case, the values for $Q = \pm 1$ are themselves the first terms of an elliptic series.

For $Q = \pm 1$ we make the following transformations of the integral (A4): The substitution $x' = x - (R/2)$ allows us to write the integral as a linear combination of integrals of the form

$$\int_{-\infty}^{\infty} \frac{x'^{2n}\,dx'}{\left\{\left[1 + \left(x' - \frac{R}{2}\right)^2\right]\left[1 + \left(x' + \frac{R}{2}\right)^2\right]\right\}^{m/2}}. \quad \text{(A6)}$$

Then, with the substitutions

$$y^2 = x', \quad y = \left(1 + \frac{R^2}{4}\right)\cot^2(\zeta/2),$$

this becomes

$$\int_0^{\infty} \frac{y^{n-\frac{1}{2}}\,dy}{\left\{\left[y + \left(1 - \frac{R^2}{4}\right)\right]^2 + R^2\right\}^{m/2}} = \left(1 + \frac{R^2}{4}\right)^{n-m+\frac{1}{2}} \int_0^{\pi} \frac{\left(\cos^{2n}\frac{\zeta}{2}\right)\sin^{2m-2n-2}\frac{\zeta}{2}\,d\zeta}{\Delta^m}, \quad \text{(A7)}$$

where

$$\Delta = (1 - k^2 \sin^2\zeta)^{\frac{1}{2}}, \quad k^2 = R^2/(R^2 + 4).$$

The integrals needed to evaluate the terms in $V(r)$ have $m = 3$, and the relevant elliptic integrals may be found in Ref. 4, p. 56. One finds

$$\int \theta_0(1)\theta_0^5(2)\, d^3r = \pm \frac{4\pi}{3g^{\frac{3}{2}}}\left[\frac{B(k)}{(R^2+4)^{\frac{1}{2}}} + \frac{2D(k)}{(R^2+4)^{\frac{3}{2}}}\right],$$

$$\int \theta_0^3(1)\theta_0^3(2) d^3r = \pm \frac{8\pi}{g^{\frac{3}{2}}}\frac{D(k)}{(R^2+4)^{\frac{3}{2}}}. \qquad \text{(A8)}$$

To evaluate the integral in Eq. (12), we first note that

$$\nabla_\alpha\nabla_\beta\theta_0 = \frac{3z(r_\beta - \xi_\beta)(r_\alpha - \xi_\alpha)}{[z^4g + (\mathbf{r} - \boldsymbol{\xi})^2]^{\frac{5}{2}}} - \frac{z\delta_{\alpha\beta}}{[z^4g + (\mathbf{r} - \boldsymbol{\xi})^2]^{\frac{3}{2}}}. \qquad \text{(A9)}$$

Again, making the substitutions (A2), we find

$$T_{12} = -\frac{Q}{|Q|}\sum_{\alpha,\beta}\dot{\xi}_{1\alpha}\dot{\xi}_{2\beta}\frac{1}{g^{\frac{1}{2}}}\int\frac{d^3x}{[Q^{-2} + (\mathbf{x}+\mathbf{R})^2]^{\frac{1}{2}}} \times\left[\frac{3x_\alpha x_\beta}{(Q^2+x^2)^{\frac{5}{2}}} - \frac{\delta_{\alpha\beta}}{(Q^2+x^2)^{\frac{3}{2}}}\right]$$

$$\equiv \sum_{\alpha,\beta}\dot{\xi}_{1\alpha}\dot{\xi}_{2\beta}A_{\alpha\beta}. \qquad \text{(A10)}$$

We again specialize to the case $Q = \pm 1$. The $\delta_{\alpha\beta}$ part of the integral is of the form of the integral in (A3) and is readily evaluated by the above method. The first integral in (A10) vanishes for $\alpha \neq \beta$ when the azimuthal integration is performed. If the polar axis, about which there is cylindrical symmetry, is chosen to be in the z direction, then $A_{11} = A_{22}$. We also observe that

$$\mathrm{Tr}\, A_{\alpha\beta} = \mp\frac{3}{g^{\frac{1}{2}}}\int\frac{d^3x}{[1+(\mathbf{x}+\mathbf{R})^2]^{\frac{1}{2}}}\times\left[\frac{x^2}{(1+x^2)^{\frac{5}{2}}} - \frac{1}{(1+x^2)^{\frac{3}{2}}}\right]$$

$$= \pm\frac{3}{g^{\frac{1}{2}}}\int\frac{d^3x}{[1+(\mathbf{x}+\mathbf{R})^2]^{\frac{1}{2}}(1+x^2)^{\frac{5}{2}}}$$

$$= 3g\int\theta_0(1)\theta_0^5(2)\, d^3r$$

$$= \pm\frac{4\pi}{g^{\frac{1}{2}}}\left[\frac{B(k)}{(R^2+4)^{\frac{1}{2}}} + \frac{2D(k)}{(R^2+4)^{\frac{3}{2}}}\right]. \qquad \text{(A11)}$$

It is therefore sufficient to evaluate

$$A_{33} = \pm\frac{1}{g^{\frac{1}{2}}}\int\frac{d^3x}{(1+x^2+R^2+2xR\cos\theta)^{\frac{1}{2}}}\times\left[\frac{3x^2\cos^2\theta}{(1+x^2)^{\frac{5}{2}}} - \frac{1}{(1+x^2)^{\frac{3}{2}}}\right]. \qquad \text{(A12)}$$

It remains to evaluate

$$\frac{3\times 2\pi}{g^{\frac{1}{2}}}\int_0^\infty dx\int_{-1}^1\frac{d(\cos\theta)x^4\cos^2\theta}{(1+x^2+R^2+2xR\cos\theta)^{\frac{1}{2}}(1+x^2)^{\frac{5}{2}}}$$

$$= \frac{6\pi}{g^{\frac{1}{2}}}\int_0^\infty\frac{x^4\, dx}{(1+x^2)^{\frac{5}{2}}(1+x^2+R^2)^{\frac{1}{2}}}\int_{-1}^1\frac{p^2\, dp}{(1+tp)^{\frac{1}{2}}}, \qquad \text{(A13)}$$

where

$$t = 2xR/(1+x^2+R^2).$$

The integral

$$\int_{-1}^1\frac{p^2\, dp}{(1+tp)^{\frac{1}{2}}} = \frac{2}{15t^3}(8 - 4tp + 3t^2p^2)(1+tp)^{\frac{1}{2}}\Big|_{-1}^1 \qquad \text{(A14)}$$

remains finite for $x \to \pm\infty$, $t \to 0$, since, if we expand this expression for small tp, we get

$$\left[\frac{16}{15t^3} + \frac{p^3}{3} + O(tp^4)\right]_{-1}^1 = \left[\frac{p^3}{3} + O(tp^4)\right]_{-1}^1.$$

After combining the contributions from the upper and lower limits of (A14), the integral (A13) becomes

$$\frac{4\pi}{5g^{\frac{1}{2}}}\int_{-\infty}^\infty\frac{x^4\, dx}{(1+x^2)^{\frac{5}{2}}(1+x^2+R^2)^{\frac{1}{2}}}\left[\frac{(1+x^2+R^2)^3}{x^3R^3} - \frac{(1+x^2+R^2)^2}{x^2R^2} + \frac{3}{2}\frac{(1+x^2+R^2)}{xR}\right]. \qquad \text{(A15)}$$

After considerable labor, this may be reduced to a collection of integrals of the form (A6) with values of m being 1, 3, and 5. All of these may be found by the above methods or in tables.[4,5] The results simplify to give

$$A_{33} = \pm\frac{8\pi}{g^{\frac{1}{2}}}\frac{D(k)}{(R^2+4)^{\frac{3}{2}}},\quad A_{11} = \pm\frac{2\pi}{g^{\frac{1}{2}}}\frac{B(k)}{(R^2+4)^{\frac{1}{2}}}. \qquad \text{(A16)}$$

The expression for T_{12} can be cast into a rotationally invariant form by writing

$$T_{12} = A_{11}(\dot{\xi}_{1x}\dot{\xi}_{2x} + \dot{\xi}_{1y}\dot{\xi}_{2y}) + A_{33}\dot{\xi}_{1z}\dot{\xi}_{2z}$$

$$= A_{11}\dot{\boldsymbol{\xi}}_1\cdot\dot{\boldsymbol{\xi}}_2 + (A_{33} - A_{11})\dot{\xi}_{1z}\dot{\xi}_{2z}$$

$$= A_{11}\dot{\boldsymbol{\xi}}_1\cdot\dot{\boldsymbol{\xi}}_2 + (A_{33} - A_{11})\frac{R_iR_j}{R^2}\dot{\xi}_{1i}\dot{\xi}_{2j}. \qquad \text{(A17)}$$

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[5] W. Grobner and N. Hofreiter, *Integraltafel* (Springer–Verlag, Berlin, 1961), Vol. I.